The Method of Differences

The method of differences is a "sneaky" trick whereby the sum of a series is established under certain conditions, and a great deal of "cancelling out" of terms contributes to a rather "slick" method.

THE METHOD OF DIFFERENCES can be applied to find the sum of a series so **long as the general term in the series can be expressed in the form**

$$f(k+1)-f(k)$$

where *f( )* is some function.

This is particularly useful when the general term of the series is that of an **ALGEBRAIC FRACTION** that can be split into the **DIFFERENCE** of two fractions by the method of **PARTIAL FRACTIONS**.

EXAMPLE 1

FIND $\sum_{r=3}^{n}\frac{1}{(r+1)(r+2)}$ **by first expressing the sum in PARTIAL FRACTIONS.**

Using Partial Fractions let

$$\frac{1}{(r+1)(r+2)}\equiv\frac{A}{(r+1)}+\frac{B}{(r+2)}$$

$$\frac{1}{(r+1)(r+2)}\equiv\frac{A(r+2)+B(r+1)}{(r+1)(r+2)}$$

$$1\equiv A(r+2)+B(r+1)$$

Let r=-1

1=A(-1+2)
A=1

Let r=-2

1=B(-2+1)
B=-1

$$\frac{1}{(r+1)(r+2)}\equiv\frac{1}{(r+1)}-\frac{1}{(r+2)}$$

APPLYING THE METHOD OF DIFFERENCES JUST MEANS SUBSTITUTING THE VALUES FOR r AND SUMMING (NICE THINGS HAPPEN IN CANCELLING)

now summing from r=3 to r=n

$$\frac{1}{(r+1)(r+2)} \equiv \frac{1}{(r+1)} - \frac{1}{(r+2)}$$

$$\frac{1}{(r+1)(r+2)} \equiv \frac{1}{(r+1)} - \frac{1}{(r+2)} \equiv \frac{1}{4} - \cancel{\frac{1}{5}}$$

$$+\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}$$

$$+\cancel{\frac{1}{6}} - \cancel{\frac{1}{7}}$$

$$+\cancel{\ldots\ldots\ldots\ldots}$$

$$+\cancel{\ldots\ldots\ldots\ldots}$$

$$+\cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}}$$

$$+\cancel{\frac{1}{n+1}} - \frac{1}{n+2}$$

The example has listed the Sum vertically for convienience but horizontally is fine so long as brackets are effectively used.

NOTICE THAT MANY PARTS OF THE SUM WILL CANCEL OUT. THIS IS THE "TRICK" INVOLVED IN THE METHOD OF DIFFERENCES

The sum becomes (after cancelling of terms with opposite signs)

$$\sum_{r=3}^{n} \frac{1}{(r+1)(r+2)} = \frac{1}{4} - \frac{1}{(n+2)}$$

This can be expressed as a single fraction as:

$$\sum_{r=3}^{n} \frac{1}{(r+1)(r+2)} = \frac{1(n+2)-4}{4(n+2)}$$

$$= \frac{n-2}{4(n+2)}$$

This is the sum as a single fraction

EXAMPLE 2

Find $\sum_{r=1}^{n} \frac{1}{r(r+2)}$

Using Partial Fractions let

$$\frac{1}{r(r+2)} \equiv \frac{A}{r} + \frac{B}{(r+2)}$$

$$\frac{1}{r(r+2)} \equiv \frac{A(r+2)+Br}{r(r+2)}$$

$$1 \equiv A(r+2)+Br$$

let r=-2
1=B(-2)

$B = \frac{-1}{2}$

Let r=0
1=A(2)

$A = \frac{1}{2}$

$$\frac{1}{r(r+2)} \equiv \frac{1}{2r} - \frac{1}{2(r+2)}$$

Now sum from r=1 to n and look for terms that cancel (You will have to look a little harder this time!)

First we will take the half out as a common factor to make things easier

$$\frac{1}{r(r+2)} \equiv \frac{1}{2r} - \frac{1}{2(r+2)} \equiv \frac{1}{2}\left(\frac{1}{r} - \frac{1}{(r+2)}\right)$$

$$\frac{1}{r(r+2)} \equiv \frac{1}{2}\left(\frac{1}{r} - \frac{1}{(r+2)}\right) \equiv \frac{1}{2}\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right.$$

$$+\frac{1}{2} - \cancel{\frac{1}{4}}$$

$$+\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}$$

$$+\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}$$

$$+\cancel{\ldots\ldots\ldots}$$

$$+\cancel{\ldots\ldots\ldots}$$

$$+\cancel{\frac{1}{n-2}} - \cancel{\frac{1}{n}}$$

$$+\cancel{\frac{1}{n-1}} - \frac{1}{n+1}$$

$$\left.+\cancel{\frac{1}{n}} - \frac{1}{n+2}\right)$$

This time there is a jump of TWO before we see the fractions cancel. This leaves us with TWO at the start and TWO at the end.

$$\frac{1}{r(r+2)} \equiv \frac{1}{2}\left(\frac{1}{r} - \frac{1}{(r+2)}\right) \equiv \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$$

The TWO at the start

The TWO at the end

Finally Simplifying and multiplying the half:

$$\frac{1}{r(r+2)} \equiv \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$$

EXAMPLE 3

Find $\sum_{r=1}^{n} \frac{1}{r(r+1)}$

1997 OLD P4

Show that

$$\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} \equiv \frac{2}{r(r+1)(r+2)}$$

Hence , or otherwise find a simplified expression for

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$$

(NOTE: Thre is no need for Partial Fractions as we must show the difference between the two fractions is the SINGLE fraction given. Partial fractions would be needed if working the other way around. ie show that the single fraction can be written as the difference between the two fractions.)

These questions are not as frequently examined as some topics but be aware of them! (2008 Jan was the last time)

_P4 Modular 2004 (7 marks)

Show that

$$\frac{1}{1\times 3}+\frac{1}{2\times 4}+\frac{1}{3\times 5}+.....+\frac{1}{n(n-2)}=\frac{3}{4}-\frac{(2n+3)}{2(n+1)(n+2)}$$

YOU SHOULD NOTICE THAT THIS IS EXAMPLE 2 BUT WITHOUT THE SIGMA NOTATION

1999 OLD P4 MODULAR (5 marks)

Express $\dfrac{2}{4r^2-1}$ in partial fractions.

Show that

$$\sum_{r=1}^{n}\frac{2}{4r^2-1}=\frac{2n}{2n+1}$$

NOTICE THAT WE HAVE THE DIOFFERENCE OF TWO SQUARES ON THE DENOMINATOR
SIMILAR QUESTION TO 2008 JAN!

USING THE METHOD OF DIFFERENCES TO PROVE THE STANDARD FORMULAE IN THE FORMULAE BOOKLET

The following results are all found in the formula booklet of the WJEC examinations but can all be proved bt using "THE METHOD OF DIFFERENCES"

$$\sum_{r=1}^{n} r \equiv \frac{n}{2}(n+1)$$

$$\sum_{r=1}^{n} r^2 \equiv \frac{n}{6}(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 \equiv \frac{n^2}{4}(n+1)^2$$

These can all been proved by using the **METHOD OF DIFFERENCES**.

The method of differences can only be used if the series can be expressed as $f(r+1)-f(r)$ or equivalent.

To find the sum of the natural numbers we used the fact

$$(k+1)^2 - k^2 \equiv 2k+1$$

These are the SAME function
Ie $(....)^2$.
The value of the variable in the first is one more than the value of the variable in the second.
We are finding the DIFFERENCE.

To find the sum of the squares of the positive integers we used the fact that

$$(k+1)^3 - k^3 \equiv 3k^2 + 3k + 1$$

These are the SAME function
Ie $(....)^3$.
The value of the variable in the first is one more than the value of the variable in the second.
We are finding the DIFFERENCE.

The same applied to the sum of the cubes of the positive integers when we used the fact that

$$(k+1)^4 - k^4 \equiv 4k^3 + 6k^2 + 4k + 1$$

The process involved substituting n=1,2,3,…….(n-1),n into both sides of the "DIFFERENCE EQUATION". Then they were SUMMED, resulting in some convenient cancelling and an equation that could be rearranged to obtain the desired sum.