POLAR COORDINATES

WJEC REQUIREMENTS

The candidates are required to sketch simple curves expressed in polar form.
SOME Examples **could** be:

$r = a$

$\theta = \alpha$

$r = a\theta$

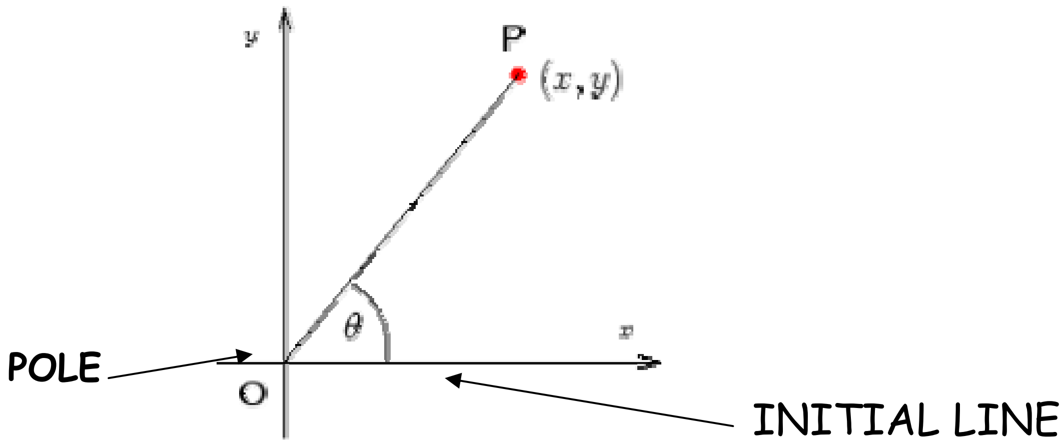
$r = a\sin^2 n\theta$

Where r is a variable distance of a point from a POLE, as it rotates anticlockwise an angle θ about the POLE, starting from an INITIAL LINE. α and n are constants.

There are THREE main problem types that candidates are required to tackle:

1. Intersection of TWO curves.
2. Finding points where tangents are either parallel or Perpendicular to the initial line.
3. Calculations of area as the radius vector moves from one angle to another.

We will look at each of these in turn.



Polar coordinates are a system of coordinates that locate points on a plane by defining the distance from the origin (or the POLE) and the angle of rotation (from the INITIAL LINE).

The **Polar Coordinates** of P are (r, θ) where $r = OP > 0$ and θ is the angle from OX round to OP **<u>anticlockwise.</u>**

By convention, we take the angle range to be $-\pi \le \theta < \pi$ or sometimes $0 \le \theta < 2\pi$ will be used.

CARTESIAN AND POLAR COORDINATES

Consider the problem of converting between cartesian and polar coordinates.

One way round is easy: from the above diagram:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

and so

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

we must however be careful about the case when x=0 and select the quadrant depending on signs.

if $x = 0$ then if $y > 0$ then $\theta = \pi/2$ if $y < 0$ then $\theta = -\pi/2$

You will be expected to sketch simple curves given in polar form.

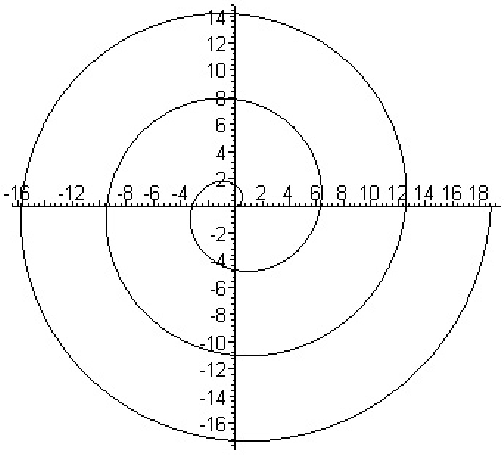
To assist with the recognition of standard curves, visit www.mathsnet.net and find the A level section FP3. Here you will be able to generate sketches of various standard curves (such as cardiods)

You can see many visual investigations into different families of polar graphs on

http://mathdemos.gcsu.edu/mathdemos/family_of_functions/polar_gallery.html

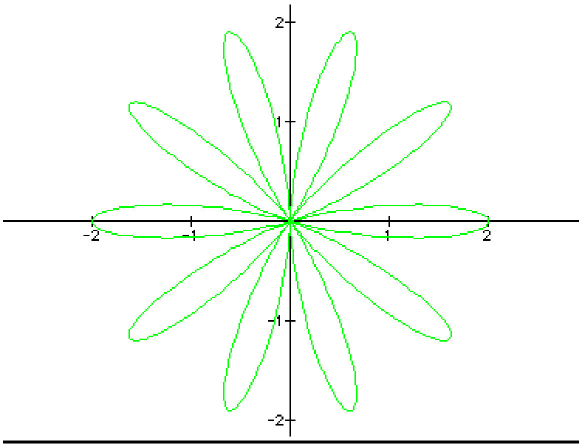
All very pretty!!!!! Like spirograph.

SPIRALS



Graph of $r = \theta$
(archimedean spiral)

ROSE CURVES



Graph of $r = 1 + \cos(10\theta)$

(rose curve with 10 "petals")

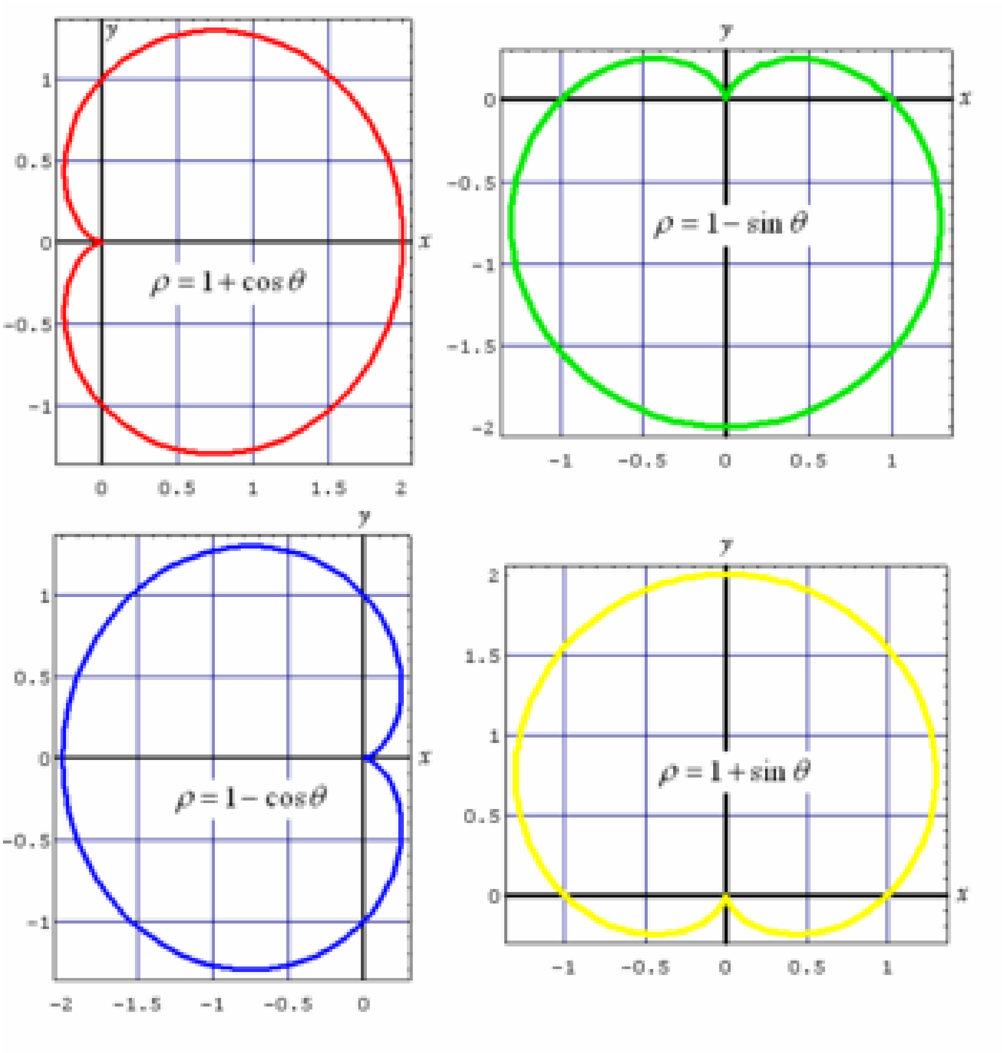
Curves with the rose petal shape come in the form

$$r = a\cos(n\theta) \text{ and } r = a\sin(n\theta).$$

Circles within polar coordinates are created with equations of the form

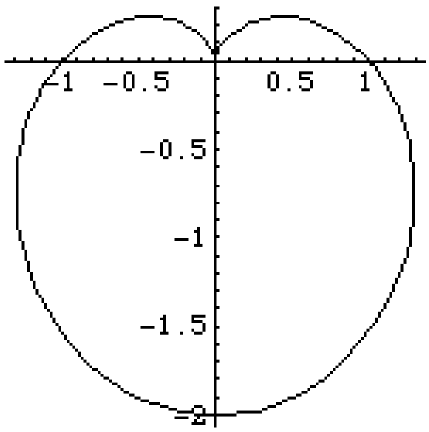
$$r = a, \quad r = a\cos\theta, \text{ and } r = a\sin\theta.$$

CARDIODS

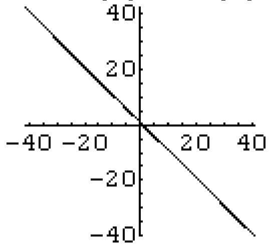


Cardioids are made by graphs with equations such as

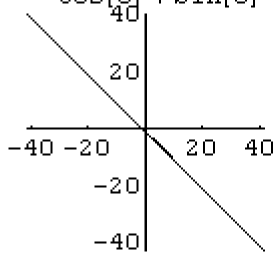
$$r = a + b\cos\theta \text{ and } r = a + b\sin\theta.$$



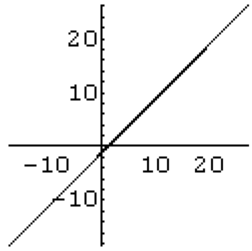
$r=\frac{1}{\cos[\theta]+\sin[\theta]}$



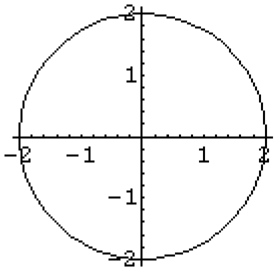
$r=\frac{-1}{\cos[\theta]+\sin[\theta]}$



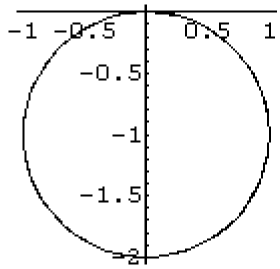
$r=\frac{1}{\cos[\theta]-\sin[\theta]}$



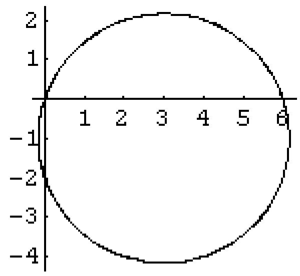
$r=-2$



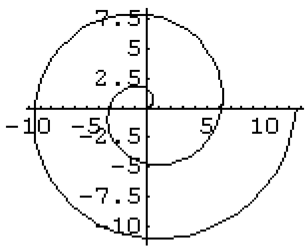
$r=-2\sin[\theta]$



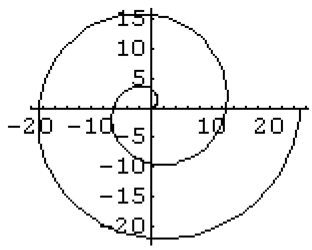
$r=6\cos[\theta]-2\sin[\theta]$



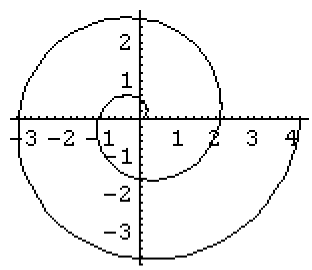
$r=\theta$



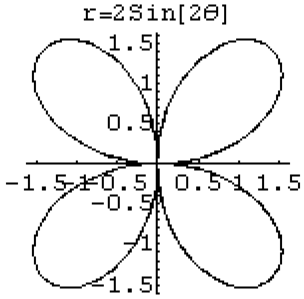
$r=2\theta$



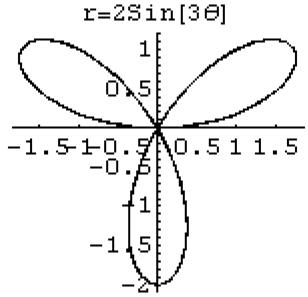
$r=\theta/3$



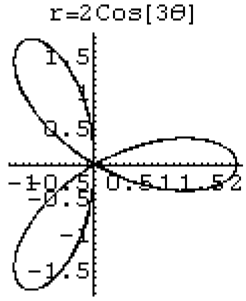
$r=2\sin[2\theta]$

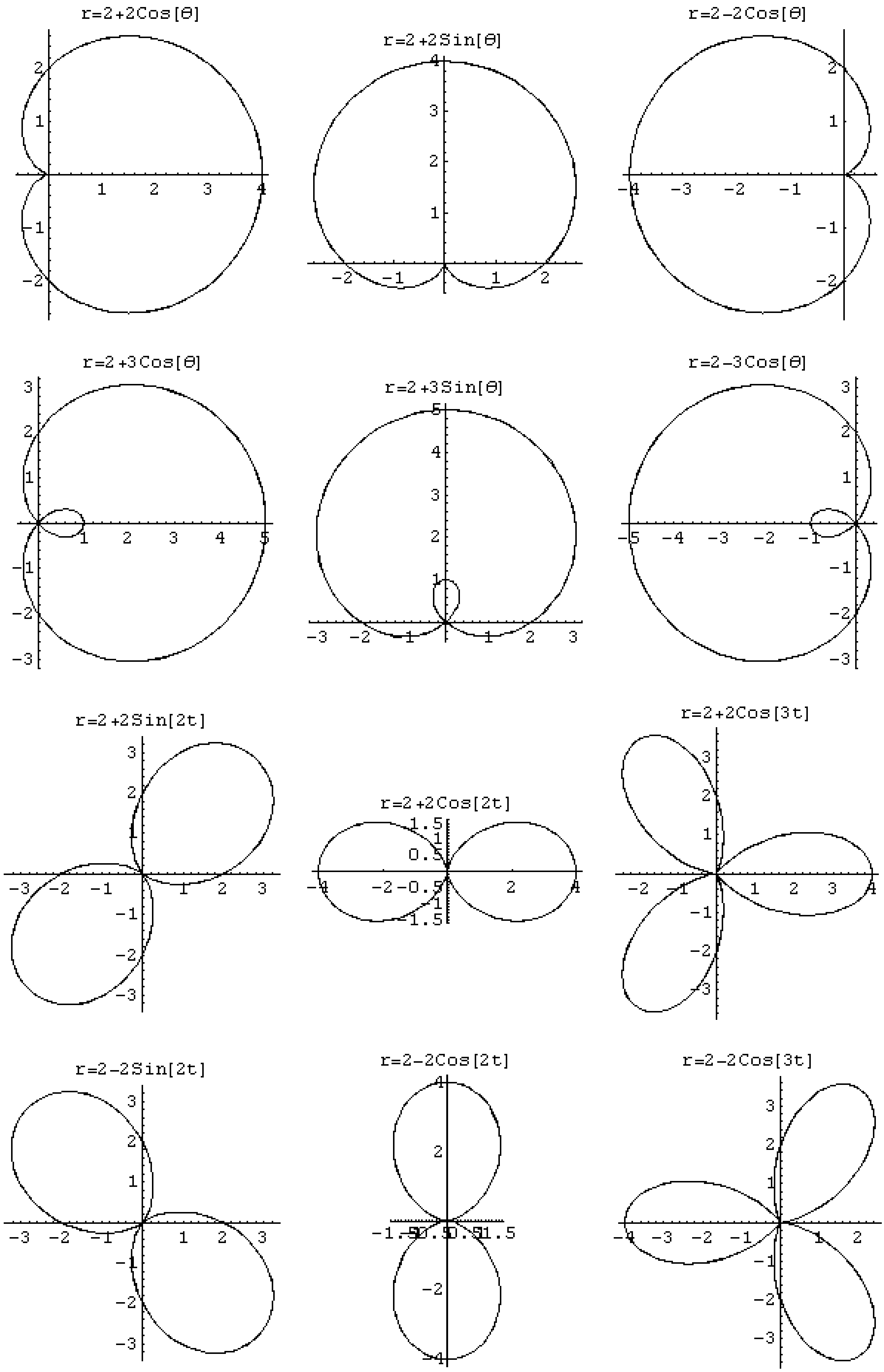


$r=2\sin[3\theta]$



$r=2\cos[3\theta]$




r=2+2Cos[θ]
r=2+2Sin[θ]
r=2-2Cos[θ]
r=2+3Cos[θ]
r=2+3Sin[θ]
r=2-3Cos[θ]
r=2+2Sin[2t]
r=2+2Cos[2t]
r=2+2Cos[3t]
r=2-2Sin[2t]
r=2-2Cos[2t]
r=2-2Cos[3t]

INTERSECTION OF TWO CURVES

If $r = f_1(\theta)$ and

$r = f_2(\theta)$

then the points of intersection are clearly found by equating the functions.

Thus we will have an equation in θ which will be solved and the corresponding values of r then located.

EXAMPLE

If $r = 1 - \cos\theta$ AND

$r = 3\sin^2\theta$ then the points of intersection are found by equating.

$$1 - \cos\theta = 3\sin^2\theta$$

and solving

$$0 = 3\sin^2\theta + \cos\theta - 1$$

$$0 = 3 - 3\cos^2\theta + \cos\theta - 1$$

$$3\cos^2\theta - \cos\theta + 1 - 3 = 0$$

$$3\cos^2\theta - \cos\theta - 2 = 0$$

$$(3\cos\theta + 2)(\cos\theta - 1) = 0$$

$$(3\cos\theta + 2) = 0$$

$$\cos\theta = \frac{-2}{3}$$

$$(\cos\theta - 1) = 0$$

$$\cos\theta = 1$$

Thus θ=0, 2.3, 3.98 radians

Substituting to find r will give the three points of intersection

$$(r, \theta) = (0,0), (\frac{5}{3}, 2.30), (\frac{5}{3}, 3.98)$$

POINTS WHERE TANGENTS ARE EITHER PARALLEL OR PERPENDICULAR TO THE INITIAL LINE.

These are points where $y = r\sin\theta$ or $x = r\cos\theta$ take on STATIONARY VALUES.

EXAMPLE

Find the points on the Cardiod $r = 1+\cos\theta$ at which the tangent is PARALLEL to the initial line.

SOLUTION

Consider $y = r\sin\theta$

Substituting $r = 1+\cos\theta$ **we get**

$$y = (1+\cos\theta)\sin\theta$$

$$y = \sin\theta + \cos\theta\sin\theta$$

Differentiating With respect to theta:

$$\frac{dy}{d\theta} = \cos\theta + \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta + \cos\theta - 1$$

$$= (2\cos\theta - 1)(\cos\theta + 1)$$

For a STATIONARY POINT this is ZERO

$$(2\cos\theta - 1)(\cos\theta + 1) = 0$$

gives solutions

$(2\cos\theta - 1) = 0$

$\cos\theta = \frac{1}{2}$

$(\cos\theta + 1) = 0$

$\cos\theta = -1$

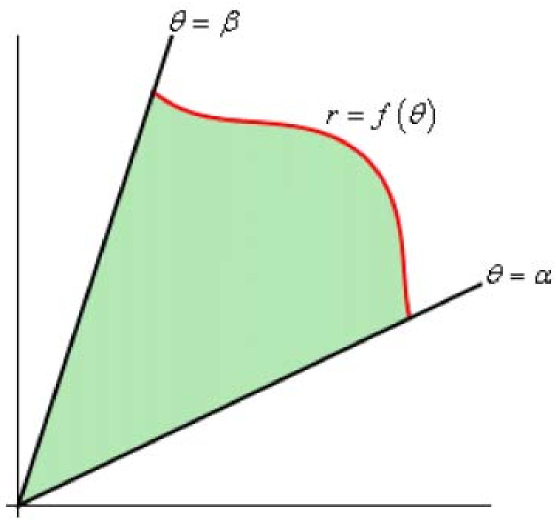
$\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ Finding the corresponding values of r by substitution gives

The POINTS AT WHICH THE TANGENT IS PARALLEL TO THE INITIAL LINE ARE $(\frac{3}{2}, \frac{\pi}{3}), (0, \pi), (\frac{3}{2}, \frac{5\pi}{3})$

IF THE SOLUTION WAS REQUIRED AS PERPENDICULAR TO THE INITIAL LINE WE WOULD START WITH $x = r\cos\theta$ and repeat the method

Area with Polar Coordinates

In this section we are going to look at areas enclosed by polar curves. Note as well that we said "enclosed by" instead of "under" as we typically have in these problems. These problems work a little differently in polar coordinates. Here is a sketch of what the area that we'll be finding in this section looks like.



We'll be looking for the shaded area in the sketch above. The formula for finding this area is,

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

Notice that we use r in the integral instead of $f(\theta)$ so make sure and substitute accordingly when doing the integral.

Let's take a look at an example.

Example 1 Determine the area of the inner loop of $r = 2 + 4\cos\theta$.

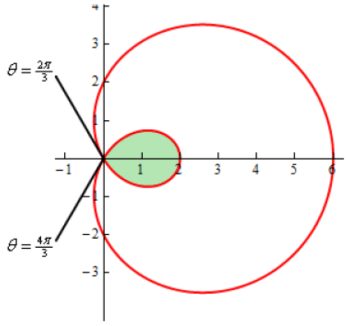
Solution

We graphed this function back when we first started looking at polar coordinates. For this problem we'll also need to know the values of θ where the curve goes through the origin. We can get these by setting the equation equal to zero and solving.

$$0 = 2 + 4\cos\theta$$

$$\cos\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Here is the sketch of this curve with the inner loop shaded in.



Can you see why we needed to know the values of θ where the curve goes through the origin? These points define where the inner loop starts and ends and hence are also the limits of integration in the formula.

So, the area is then,

$$A = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(2+4\cos\theta\right)^2 d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(4+16\cos\theta+16\cos^2\theta\right)d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 2+8\cos\theta+4\left(1+\cos\left(2\theta\right)\right)d\theta$$

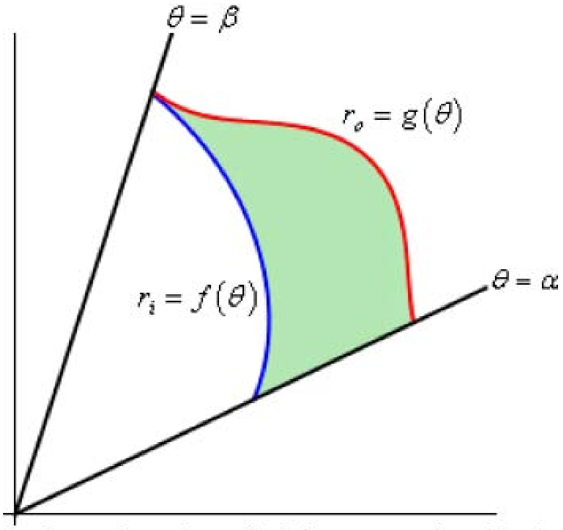
$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 6+8\cos\theta+4\cos\left(2\theta\right)d\theta$$

$$= \left(6\theta+8\sin\theta+2\sin\left(2\theta\right)\right)\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}$$

$$= 4\pi - 6\sqrt{3} = 2.174$$

AREAS BETWEEN TWO CURVES

So, that's how we determine areas that are enclosed by a single curve, but what about situations like the following sketch were we want to find the area between two curves.

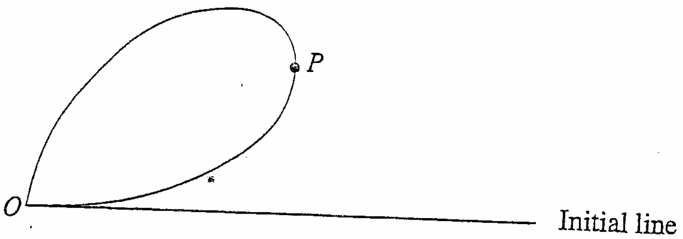


In this case we can use the above formula to find the area enclosed by both and then the actual area is the difference between the two. The formula for this is,

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_o^2 - r_i^2\right) d\theta$$

PAST PAPER QUESTIONS. (POLAR COORDINATES FP3 and P6)
2002 P6

4.



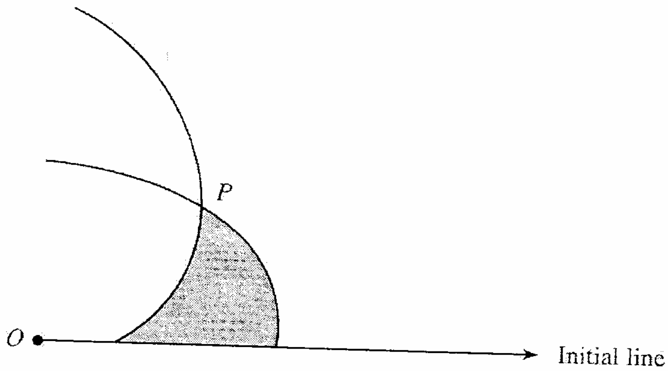
The diagram shows the curve C with polar equation $r = \sin 2\theta$ $(0 \leqslant \theta \leqslant \frac{\pi}{2})$.

(a) Show that the area of the region enclosed by C is equal to $\frac{\pi}{8}$. [4]

(b) The tangent to C at the point P is perpendicular to the initial line. Find the θ-coordinate of P, giving your answer in radians correct to three significant figures. [8]

PAST PAPER QUESTIONS. (POLAR COORDINATES FP3 and P6)
2003 P6

3.



The above diagram shows the curves with polar equations

$$r = 2e^{-\theta} \qquad (0 \leqslant \theta \leqslant \tfrac{\pi}{2}),$$

$$r = \tfrac{1}{2}e^{\theta} \qquad (0 \leqslant \theta \leqslant \tfrac{\pi}{2}),$$

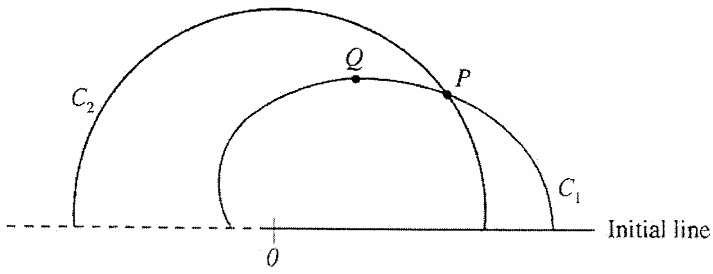
intersecting at the point P.

(a) Find the polar coordinates of P. [5]

(b) Find the exact value of the area of the shaded region. [6]

PAST PAPER QUESTIONS. (POLAR COORDINATES FP3 and P6)
2004 P6

8.



The above diagram shows the curves with polar equations

$$C_1 : r = 3 + 2\cos\theta$$
$$C_2 : r = 4$$

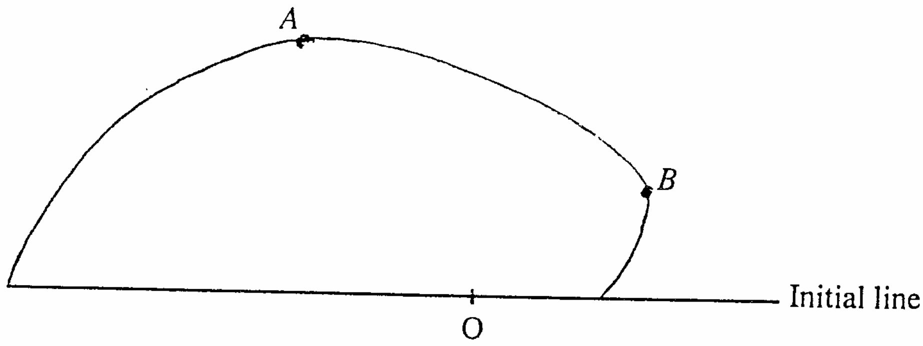
defined for $0 \leqslant \theta \leqslant \pi$.

(a) Find the polar coordinates of P, the point of intersection of C_1 and C_2. [4]

(b) Find the polar coordinates of Q, the point at which the tangent to C_1 is parallel to the initial line. [8]

PAST PAPER QUESTIONS.(P6) Further Specimen

5.



The figure (not drawn accurately) shows part of the curve with polar equation $r = e^{\theta}$.

(a) Find the coordinates of the points A and B, where the tangents to the curve are respectively parallel to, and perpendicular to, the initial line. [8]

(b) Find the area enclosed between the curve and the initial line. [4]

(c) Find the polar equation of the tangent to the curve at B. [3]

PAST PAPER QUESTIONS. (POLAR COORDINATES FP3 and P6)
Specimen FP3 (2005/6)

6. The curves C_1 and C_2 have polar equations as follows:

$$C_1 : r = 1 - \cos\theta \qquad (-\pi \leq \theta \leq \pi)$$
$$C_2 : r = \cos 2\theta \qquad (-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4})$$

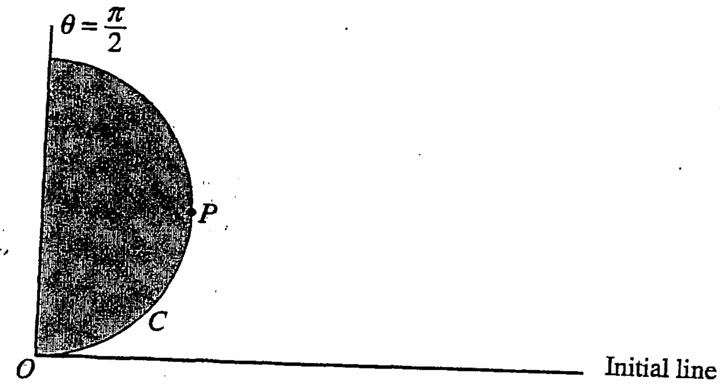
(a) Sketch C_1 and C_2 on the same diagram. [2]

(b) Find the area enclosed by C_1. [5]

(c) Find the polar coordinates of the points of intersection of C_1 and C_2. [6]

PAST PAPER QUESTIONS 2006. (POLAR COORDINATES FP3)

6.



The diagram shows the initial line, the line $\theta = \frac{\pi}{2}$ and the curve C with equation

$$r = \sinh\theta \ (0 \leqslant \theta \leqslant \frac{\pi}{2}).$$

(a) Find the area of the shaded region. [5]

(b) The tangent to C at the point P is perpendicular to the initial line.

(i) Show that the θ coordinate of P satisfies the equation

$$\tanh\theta = \cot\theta.$$

(ii) Starting with the initial approximation $\theta_0 = 1$ to the root of this equation, use the Newton-Raphson method **once** to find a better approximation θ_1. Give your answer correct to four significant figures. [10]

Last part of this requires Newton Raphson Method for a numerical method to find the solution of equations.