1. The function f is defined by

$$f(x) = x^2 + \frac{a}{x}, \qquad \text{for } 0 < x \leqslant 1,$$

$$f(x) = bx + \frac{1}{2}x^3 - 1, \qquad \text{for } x > 1,$$

where a and b are constants. Given that both f and its derivative are continuous at $x = 1$, find the values of a and b. [6]

2. Find the general solution, in radians, of the equation

$$\sin\theta + \sin 5\theta = \sin 3\theta. \qquad [7]$$

3. *(a)* Find an expression, in its simplest form, for

$$\sum_{r=1}^{n} (2r-1)^2. \qquad [6]$$

(b) Hence evaluate

$$1^2 + 3^2 + 5^2 + \dots\dots\dots + 47^2 + 49^2. \qquad [2]$$

4. Use mathematical induction to prove that $5^{2n} + 9^n - 2$ is divisible by 8 for all positive integer values of n. [7]

5. The roots of the cubic equation

$$x^3 - 2x^2 + 3x + 4 = 0$$

are denoted by α, β, γ.

(a) Find the cubic equation whose roots are $\beta\gamma$, $\gamma\alpha$ and $\alpha\beta$. [9]

(b) Show that

$$\alpha^2 + \beta^2 + \gamma^2 = -2.$$

Hence state the number of real roots of the above cubic equation. Give a reason for your answer. [5]

6. A parabola has equation $y^2 = 4ax$.

(a) Write down the equation of the line that has gradient m and passes through the point $(2a, 0)$. [1]

(b) This line meets the parabola at the points P and Q. The mid-point of PQ is denoted by R. Show that the y-coordinate of R is $\frac{2a}{m}$, and find an expression, in terms of a and m, for the x-coordinate of R. [7]

(c) Show that as m varies, the locus of R is a parabola. [3]

(d) For this parabola, find the coordinates of its focus and the equation of its directrix. [4]

7. The function f is defined on the domain $x > 0$ by

$$f(x) = \frac{x}{2} + \frac{2}{x}.$$

(a) (i) Write down an expression for $f'(x)$.

(ii) Hence determine whether or not f is monotonic. [4]

(b) (i) Find the coordinates of the stationary point on the graph of f.

(ii) State the equations of the asymptotes on the graph of f.

(iii) Sketch the graph of f. [6]

(c) The interval $\left[\frac{1}{2}, \frac{5}{2}\right]$ is denoted by A. Determine

(i) $f(A)$,

(ii) $f^{-1}(A)$. [8]