1. (a) Given that

$$f(x) = (x - \alpha)^2(x - \beta),$$

show that $(x - \alpha)$ is a factor of $f'(x)$. [3]

(b) Given that the equation

$$27x^3 - 27x^2 + 4 = 0$$

has a repeated root, use the result given in (a) to solve this equation. [6]

2. Given the equation

$$\sin x + 3\cos x = 2,$$

use the substitution $t = \tan \dfrac{x}{2}$ to show that

$$5t^2 - 2t - 1 = 0.$$

Hence find the general solution, in radians, of the above trigonometric equation. [9]

3. Use mathematical induction to prove that

$$\sum_{r=1}^{n} r \times 2^r = (n - 1)2^{n+1} + 2$$

for all positive integer values of n. [7]

4. (a) Find an expression, in its simplest form, for

$$\sum_{r=1}^{n} r(r - 2).$$ [4]

(b) Given that the sum of the first n terms of a series is $n(n - 2)$, obtain an expression for the nth term of the series. [4]

5. The roots of the quadratic equation

$$x^2 + 2x + 4 = 0$$

are denoted by α, β. Find the cubic equation whose roots are $\dfrac{\alpha}{\beta}$, $\dfrac{\beta}{\alpha}$ and $\alpha\beta$. [13]

6. The function f is defined on the domain $x > -\frac{1}{2}$ by

$$f(x) = \frac{x + 4}{2x + 1}.$$

(a) Show that f is monotonic.

[3]

(b) (i) Sketch the graph of f. State the equations of the asymptotes.

(ii) State the range of f.

[5]

(c) Given that A is the interval $[1, 2]$, determine

(i) $f(A)$,

(ii) $f^{-1}(A)$.

[8]

7. An ellipse has equation

$$3x^2 + 4y^2 = 12.$$

(a) Find the coordinates of the foci and the equations of the directrices of this ellipse. [8]

(b) The ellipse and the line $y = x + c$ intersect at the points R and S. Find, in terms of c, the coordinates of T, the mid-point of the chord RS.

[6]

(c) Find the equation of the locus of T as c varies.

[2]