P3 2000

1. Use Simpson's Rule with five ordinates to evaluate the integral

$$\int_0^2 \sqrt{1+x^4}\,dx.$$

Show your working and give your answer correct to two decimal places. [4]

2. Given that

$$(2-i)(z+5i) = 6+7i,$$

find z in the form $a + ib$.

Find the modulus and argument of z. [5]

3. Find the general solution of

$$\cos x + \sqrt{3}\sin x = 1.$$ [4]

4. The cubic equation

$$x^3 + ax^2 + bx + c = 0$$

has roots α, β and $\alpha + \beta$. Show that

(*a*) $\alpha\beta = \dfrac{2c}{a}$, [3]

(*b*) $a^3 + 8c - 4ab = 0$. [2]

5. Write down the first three terms in the binomial expansion of

$$\left(1 + \frac{x}{16}\right)^{\frac{1}{4}}.$$

For what range of values of x is the binomial expansion valid?

Use your expansion to find an approximation to $\sqrt[4]{15}$, giving your answer correct to three decimal places. [5]

6. The variables x, y, z satisfy the equations

$$\begin{aligned} x \qquad\; - 10z &= p, \\ 3x + 4y + \;\; 2z &= q, \\ 2x + 3y + \;\; 4z &= r. \end{aligned}$$

(*a*) By first reducing the system of equations to echelon form, show that the equations are consistent when p, q, r satisfy an equation of the form

$$ap + bq + cr = 0,$$

where the values of the constants a, b, c are to be found. [3]

(*b*) Find the general solution of the equations when $p = -1$, $q = 1$, $r = 1$ and interpret your solution geometrically. [3]

7. The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & -1 \end{pmatrix}.$$

(*a*) Find the adjugate matrix of $\mathbf{A}$. [2]

(*b*) Find the matrix $\mathbf{X}$ given that

$$\mathbf{XA} = \mathbf{B}.$$ [4]

8. The points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ lie on the parabola with equation

$$y^2 = 4ax.$$

The tangents to the parabola at P and Q are perpendicular.

(*a*) Show that $pq = -1$. [3]

(*b*) The mid-point of PQ is R. Show that the Cartesian equation of the locus of R is

$$y^2 = 2a(x - a)$$

and identify this locus. [5]

9. Given that

$$y = \int_1^x (\sin t)^{\sin t}\, dt,$$

(*a*) write down $\dfrac{dy}{dx}$, [1]

(*b*) show that

$$\frac{d^2y}{dx^2} = (\sin x)^{\sin x}\,[1 + \ln(\sin x)]\cos x.$$ [3]

10. The function f has domain $x > 0$ and is defined by

$$f(x) = x + \frac{16}{x}.$$

(*a*) Draw a rough sketch of the graph of f, indicating the coordinates of its turning point. [3]

(*b*) Find $f(A)$, where A is the interval $[2,5]$. [2]

(*c*) Find $f^{-1}(B)$, where B is the interval $[10,17]$. [3]