1999

1. Use Simpson's Rule with five ordinates to find an approximate value for

$$\int_0^1 \frac{1}{1+x^3}\,dx.$$

Show your working and give your answer correct to three decimal places. [4]

2. Given that the equation

$$x^3 - 5x^2 + 3x + 2 = 0$$

has roots α, β, γ, find the cubic equation whose roots are $\alpha^2\beta\gamma$, $\alpha\beta^2\gamma$, $\alpha\beta\gamma^2$. [5]

3. (a) Differentiate $\int_0^x \sqrt{\sin t}\,dt$ with respect to x. [1]

(b) Find the derivative of $\frac{1}{x}$ from first principles. [3]

4. (a) Find, in degrees, the general solution of

$$24\cos x - 7\sin x = 15.$$ [4]

(b) Find the values of x between 0° and 180° that satisfy the equation

$$\sin x + \sin 5x = \sin 3x.$$ [4]

5. The complex numbers z_1 and z_2 are defined by

$$z_1 = 1 - i,\ z_2 = -3 - 2i.$$

(a) Represent the number $z_1 + z_2$ as a point on the Argand diagram, and find the modulus and argument of $z_1 + z_2$. [3]

(b) Given that the complex number z_3 satisfies the equation

$$z_3 = \frac{z_2}{z_1},$$

find z_3 in the form $x + iy$. [2]

6. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{pmatrix}$$

and $\mathbf{A}^T$ is the transpose of $\mathbf{A}$.

(a) Show that $\mathbf{AA}^T = k\mathbf{I}$, where $\mathbf{I}$ is the 3×3 identity matrix and k is a constant whose value is to be found. [2]

(b) Write down the matrix $\mathbf{A}^{-1}$. [1]

(c) Solve the simultaneous equations

$$\begin{aligned} 2x + y - 2z &= 1, \\ x + 2y + 2z &= 2, \\ 2x - 2y + z &= 3. \end{aligned}$$ [3]

7. (*a*) Given the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & k & -5 \\ k & -3 & 4 \\ 3 & -1 & 1 \end{pmatrix},$$

show that det $\mathbf{A} = ak^2 + bk + c$, giving the values of a, b and c. [3]

(*b*) The variables x, y, z satisfy the equations

$$\begin{aligned} 3x + ky - 5z &= 0, \\ kx - 3y + 4z &= 0, \\ 3x - y + z &= 0. \end{aligned}$$

(i) Given that the equations have solutions other than $x = 0$, $y = 0$, $z = 0$, find the two possible values of k. [2]

(ii) Using reduction to echelon form, or otherwise, find the general solution of the equations when $k = 3$. [3]

8. The point $P(3\cos t, 2\sin t)$ lies on the ellipse C whose equation is $4x^2 + 9y^2 = 36$. The normal to C at P cuts the x-axis at Q. The line through Q parallel to the y-axis cuts OP, where O is the origin, at R.

(*a*) Show that the equation of the normal to C at P is

$$2y\cos t - 3x\sin t + 5\sin t\cos t = 0.$$ [3]

(*b*) Find the locus of R in Cartesian form. [6]

9. An **odd** function f has domain $[-3, 3]$ and it is defined for $x \geqslant 0$ by

$$\begin{aligned} f(x) &= x^2, && \text{for } 0 \leqslant x \leqslant 2, \\ f(x) &= 4, && \text{for } 2 < x \leqslant 3. \end{aligned}$$

(*a*) Sketch the graph of f. [2]

(*b*) Evaluate $\int_{-1}^{3} f(x)\,\mathrm{d}x$. [4]