Logarithms.

Logarithm is another word for an index or power.

THIS IS A POWER STATEMENT

$$\text{BASE}^{\text{POWER}} = \text{NUMBER}$$

FOR EXAMPLE : We already know that;

$10^2 = 100$ This is the POWER Statement

2 is the power to which the base 10 must be raised to give 100.

OR

2 is the logarithm which, with a base 10, gives 100.

ANY POWER STATEMENT CAN BE SWOPPED AROUND TO MAKE A LOG STATEMENT

$\log_{10} 100 = 2$ This is the LOGARITHMIC statement

THE TWO STATEMENTS SAY THE SAME THING BUT REARRANGED!!!

Similarly if,

$2^3 = 8$ (Power statement)

3 is the power to which the base 2 must be raised to give 8.

$\log_2 8 = 3$ (Log statement)

OR

Logarithm is another word for power or index.

POWER FUNCTIONS are often called EXPONENTIALS.

A LOGARITHMIC FUNCTION is therefore the INVERSE (opposite) of an EXPONENTIAL function.

These statements can be abbreviated as follows;

This symbol means "IMPLIES THAT" Both ways means each statement implies that the other is true also.

$10^2 = 100 \Leftrightarrow \log_{10} 100 = 2$

$2^3 = 8 \Leftrightarrow \log_2 8 = 3$

It follows that

$\log_5 25 = 2 \quad \Leftrightarrow$

$\log_8 512 = 3 \quad \Leftrightarrow$

$\log_3 81 = 4 \quad \Leftrightarrow$

$\Leftrightarrow \quad 5^3 = 125$

$\Leftrightarrow \quad 9^{½} = 3$

$\Leftrightarrow \quad 2^5 = 32$

You will need these later in the proofs of the Laws of Logs

$\log_a x = m \quad \Leftrightarrow$

$\log_a y = n \quad \Leftrightarrow$

$\Leftrightarrow \quad a^{m+n} = xy$

$\Leftrightarrow \quad a^{m-n} = \frac{x}{y}$

$\Leftrightarrow \quad a^{nm} = x^n$

Even though we have only considered certain bases so far, **the base of a logarithm can be any positive number (a).**

Generally we can say

$$\text{If } \log_a b = c \Leftrightarrow a^c = b$$

In the C2 examination we concentrate on logarithms of a general base a, which could be ANY number. However our calculators are programmed with the base of 10 and the base of a very special number called e (more of e in the C3 examination). We will learn and prove THREE very important laws of logarithms and how these can be used to help us solve equations with the unknown variable as a POWER.

The Laws of Logarithms.

SINCE A LOGARITHM IS A POWER THE LAWS OF LOGS SHOULD BE THE SAME AS THE LAWS OF INDICES!! AND THEY ARE!

The following rules apply to the logarithm to ANY BASE Call it base a. You have to KNOW them and PROVE them!!!!!

RULE 1: $\log_a x + \log_a y = \log_a xy$

RULE 2: $\log_a x - \log_a y = \log_a \dfrac{x}{y}$

Rule 3 $\log_a x^n = n \log_a x$

The following flow diagram shows the steps to each of the three proofs of the laws of logs. Although there are three rules to prove they follow a similar format and rely on you being able to swop between log statements and power statements

START WITH A LOG STATEMENT (OR TWO LOG STATEMENTS)
Let $m = \log_a x$
And $n = \log_a y$

Use the DEFINITION OF LOGS to change the LOG STATEMENTS into POWER statements
Let $a^m = x$
And $a^n = y$

Now MULTIPLY DIVIDE or RAISE TO A POWER OF n
(depending on what you want to prove!)

Use the RULES OF INDICES relevant to the proof
(add, subtract or multiply)

Use the DEFINITION OF LOGS to change the POWER STATEMENTS BACK into LOG statements

LINK THIS TO THE INITIAL STATEMENT
And you are DONE!!!
QED

The Laws of Indices

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^m)^n = (a^n)^m = a^{m \times n}$$

Addition of logarithms (same base)

RULE 1: $\log_a x + \log_a y = \log_a xy$

<u>Proof.</u> THE PROOF IS VERY OFTEN EXAMINED

Given that x and y are positive.

Let $\log_a x = m$ and $\log_a y = n$ (first statement)

Therefore $a^m = x$ and $a^n = y$ (from the definition of logarithms).

It follows that by multiplying	$(a^m)(a^n) = xy$
Using rules of indices (Add powers)	$a^{m+n} = xy$
(From the definition of logarithms)	$m + n = \log_a xy$

But from our first statement $m=\log_a x$ and $n=\log_a y$, therefore

$$m+n = \log_a x + \log_a y = \log_a xy$$

so $\log_a x + \log_a y = \log_a xy$ **the result is proven (QED)**

PRACTICE WRITING IT OUT FOR YOURSELF

Subtraction of logs (same base)

RULE 2: $\log_a x - \log_a y = \log_a \dfrac{x}{y}$

Given that x and y are positive

<u>Proof.</u> THE PROOF IS VERY OFTEN EXAMINED!

Let $\log_a x = m$ and $\log_a y = n$ (first statement)

Therefore $a^m = x$ and $a^n = y$ (from the definition of logarithms).

It follows that by DIVIDING $\dfrac{a^m}{a^n} = \dfrac{x}{y}$

By laws of indices (subtract Powers) $a^{m-n} = \dfrac{x}{y}$

(from the definition of logarithms). $m - n = \log_a \dfrac{x}{y}$

But from the first statement

$\log_a x = m$ and $\log_a y = n$,

therefore $m - n = \mathbf{\log_a x - \log_a y}$

$m - n = \mathbf{\log_a x - \log_a y = \log_a \dfrac{x}{y}}$

so $\mathbf{\log_a x - \log_a y = \log_a \dfrac{x}{y}}$

and the result is proven (QED)

Try writing it out yourself

log of a power of x (THE POWER RULE-COME ON DOWN!!!)

Rule 3 $$\log_a x^n = n\log_a x$$

Given that x is positive

Let $\log_a x = m$ first statement

Then by definition of logs $a^m = x$ (as before!)

Now Raise both sides to the power of n (where n is nome non zero number)

By laws of indices $$x^n = (a^m)^n = a^{nm}$$

$$x^n = a^{nm}$$

now use the definition of logs to turn the power statement back to a log statement

by definition $$\log_a x^n = nm$$

using the first statement $m = \log_a x$

gives $nm = n\log_a x = \log_a x^n$

so $\log_a x^n = n\log_a x$ and the result is proven (QED)

Try writing it out yourself

USING LAWS OF LOGS TO "SIMPLIFY" or "MANIPULATE".

EXERCISE

Express as a single logarithm using appropriate laws of logs.:

1. Log3+log7
2. Log4+log5
3. 3log4
4. 2log3+3log2
5. 2log11+log1
6. log6-log3
7. 2log8-3log2
8. 3log5+4log6-3log10

WJEC PAST PAPER QUESTION 2007 JAN C2

Express as a single logarithm

$$\log_a 36+\frac{1}{2}\log_a 256-2\log_a 48$$

WJEC PAST PAPER QUESTIONS (2004 OLD P2 JANUARY)

Given that x is positive, simplify $\frac{1}{2}\log x^6+\log 3x-4\log x$

WJEC PAST PAPER QUESTIONS (2003 OLD P2 JUNE)

Given that $\log x+3\log y-\log 16=\log x^4-\log 128$

Express x in terms of y.

WJEC PAST PAPER QUESTIONS (2005 C2 JUNE)

Express $\log_{10} 2+2\log_{10} 18-\frac{3}{2}\log_{10} 36$ as a single logarithm in its simplest form

Solving Equations Using Logarithms.

We can solve equations using logs when we have the condition that the unknown is in the POWER.

EXAMPLE 1

SOLVE

$$5^{x} = 20$$

By taking the logs of both sides, we get

$$\log 5^{x} = \log 20$$

It follows that using the **POWER RULE**

$$x \log 5 = \log 20$$

We can now isolate the unknown as follows:

$$x = \frac{\log 20}{\log 5}$$

$$x = 1.861 \text{ (3dp)}$$

NOTE YOU CAN NOT CANCEL OUT THE WORD log!!!!!!!

EXAMPLE 2

Solve the Equation

$$5^{4x-3} = 8$$

correct to 3 decimal places

EXAMPLE 3

The following example does not have the unknown as a power but the equation itself involves logs. A Simple case would involve only one log, in which case we would use "ANTILOG" or in other words take or powers (or exponents) of both sides. This will depend on the base being used. If we have log (remember this means base 10) we will have 10 as the base but later on in C3 we will have ln (which means log to the base of that special number e) we will have e as the base.

SOLVE

Log(x+3)=2

We will make each side as a power with the base of 10

$$10^{\log(x+3)} = 10^2$$

BUT BY DEFINITIONTHE LEFT HAND SIDE POWER AND LOG CANCELS OUT!

$$10^{\log(x+3)} = 10^2$$
$$(x+3) = 100$$
$$x = 97$$

IF THERE ARE MORE THAN ONE LOGARITHM WE CAN NOT DO THIS UNTIL WE HAVE USED LAWS OF LOGS TO MAKE ONLY ONE LOGARITHM

EXAMPLE 4

Given that $\log_a x + \log_a (3x+4) = 2\log_a (3x-4)$

Where x is greater than $\frac{4}{3}$, FIND THE VALUE of x

EXERCISE SOLVING EQUATIONS WITH LOGS(and other PPQu's)

Solve each equation, giving your answers to 3 significant figures.

a $3^x = 12$ **b** $2^x = 0.7$ **c** $8^{-y} = 3$ **d** $4^{\frac{1}{2}x} - 0.3 = 0$

e $5^{t+3} = 24$ **f** $16 - 3^{4+x} = 0$ **g** $7^{2x+4} = 12$ **h** $5(2^{3x+1}) = 62$

i $4^{2-3x} = 32.7$ **j** $5^x = 6^{x-1}$ **k** $7^{y+2} = 9^{y+1}$ **l** $4^{5-x} = 11^{2x-1}$

ANSWERS

$x = 2.26$ $x = -0.515$ $y = -0.528$ $x = -1.74$

$t = -1.03$ $x = -1.48$ $x = -1.36$ $x = \frac{1}{3}\left(\frac{\lg 12.4}{\lg 2} - 1\right)$

$x = 0.877$

$x = -0.172$

$$x = \frac{\lg 6}{\lg 6 - \lg 5} = 9.83$$

$$y = \frac{2\lg 7 - \lg 9}{\lg 9 - \lg 7} = 6.74$$

$$x = \frac{5\lg 4 + \lg 11}{2\lg 11 + \lg 4} = 1.51$$

WJEC PAST PAPER QUESTIONS (2004 OLD P2 JUNE)

Given that $7^{2y-1} = 3^y$, find the value of y correct to three decimal places.

WJEC PAST PAPER QUESTIONS (2003 OLD P2 JANUARY)

Solve the equation

$$\log(x^2 + 14) - \log x = 2\log 3$$

WJEC PAST PAPER QUESTIONS (2005 C2 JANUARY)

Use the substitution $3^x = u$ to solve the equation

$3^{2x} - 3^{x+2} + 14 = 0$ giving your answers correct to three decimal places

A CLOSER LOOK AT POWER FUNCTIONS (OR EXPONENTIAL FUNCTIONS) and the relationship to the log function.

1. Exponential functions

Let us look more carefully at the graph of $y = 2^x$

Finding some values and plotting some points will give us an idea of what the graph looks like.

$$f(x) = 2^x$$

$$f(0) = 2^0 = 1$$

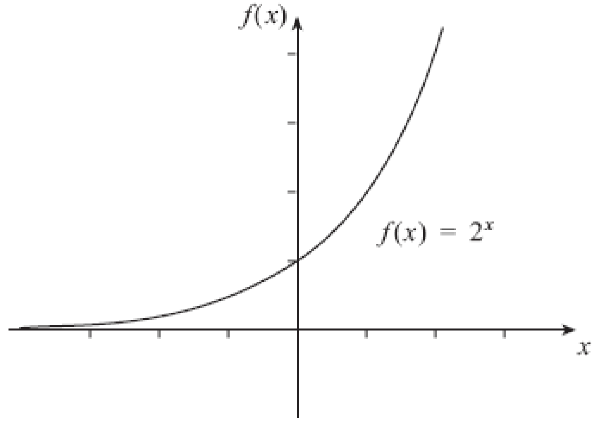
$$f(1) = 2^1 = 2 \qquad f(-1) = 2^{-1} = 1/2^1 = \tfrac{1}{2}$$

$$f(2) = 2^2 = 4 \qquad f(-2) = 2^{-2} = 1/2^2 = \tfrac{1}{4}$$

$$f(3) = 2^3 = 8 \qquad f(-3) = 2^{-3} = 1/2^3 = \tfrac{1}{8}$$

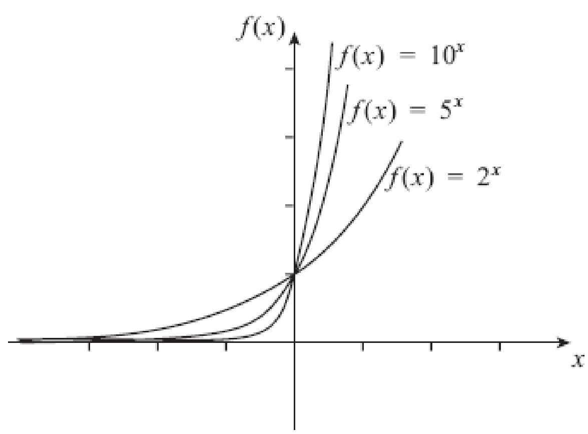
We can put these results into a table, and plot a graph of the function.

x	$f(x)$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



What about $y = 5^x$ **or** $y = 10^x$**?**

What is the effect of varying a? We can see this by looking at sketches of a few graphs of similar functions.



We see the graph simply gets steeper the larger the BASE value.

What happens if $0 < a < 1$? To examine this case, take another numerical example. Suppose that $a = \frac{1}{2}$.

$$f(x) = \left(\tfrac{1}{2}\right)^x$$

$$f(0) = \left(\tfrac{1}{2}\right)^0 = 1$$

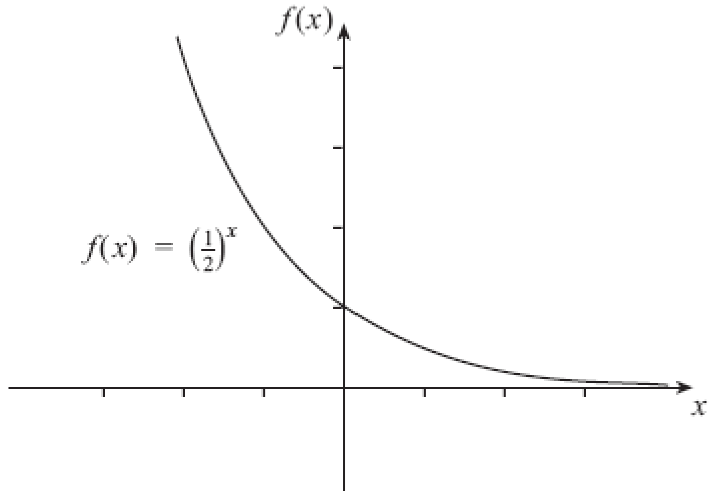
$$f(1) = \left(\tfrac{1}{2}\right)^1 = \left(\tfrac{1}{2}\right) \qquad f(-1) = \left(\tfrac{1}{2}\right)^{-1} = \left(\tfrac{2}{1}\right)^1 = 2$$

$$f(2) = \left(\tfrac{1}{2}\right)^2 = \left(\tfrac{1}{4}\right) \qquad f(-2) = \left(\tfrac{1}{2}\right)^{-2} = \left(\tfrac{2}{1}\right)^2 = 4$$

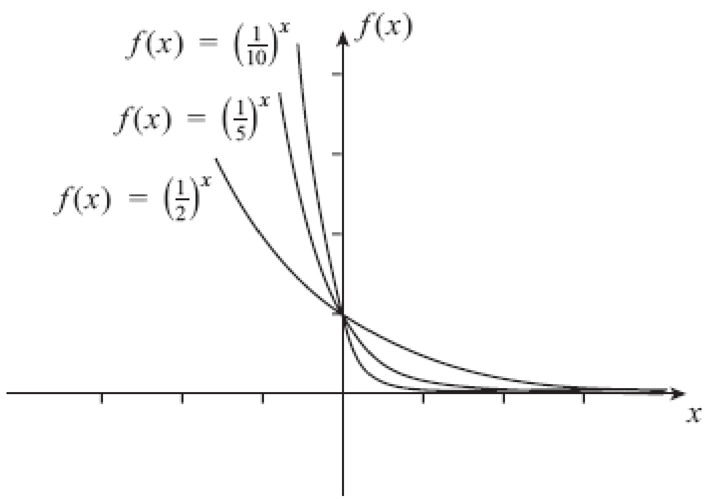
$$f(3) = \left(\tfrac{1}{2}\right)^3 = \left(\tfrac{1}{8}\right) \qquad f(-3) = \left(\tfrac{1}{2}\right)^{-3} = \left(\tfrac{2}{1}\right)^3 = 8$$

We can put these results into a table, and plot a graph of the function.

x	$f(x)$
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$



What is the effect of varying a? Again we can see by looking at sketches of a few graphs of similar functions.



So we notice that although the graphs slightly change as the value of a changes, they ALL PASS THROUGH ONE COMMON POINT!!!

This is the point (0, 1) and it is because any number to the power of zero is equal to 1.

Hence a feature of ANY EXPONENTIAL (POWER) graph is to go through the point (0,1)

Key Point

A function of the form $f(x) = a^x$ (where $a > 0$) is called an exponential function.

The function $f(x) = 1^x$ is just the constant function $f(x) = 1$.

The function $f(x) = a^x$ for $a > 1$ has a graph which is close to the x-axis for negative x and increases rapidly for positive x.

The function $f(x) = a^x$ for $0 < a < 1$ has a graph which is close to the x-axis for positive x and increases rapidly for decreasing negative x.

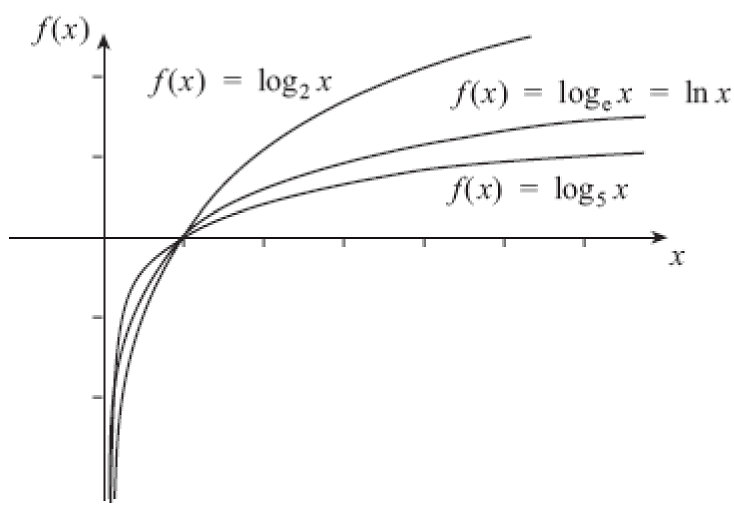
For any value of a, the graph always passes through the point $(0, 1)$. The graph of $f(x) = (1/a)^x = a^{-x}$ is a reflection, in the vertical axis, of the graph of $f(x) = a^x$.

A particularly important exponental function is $f(x) = \mathrm{e}^x$, where $\mathrm{e} = 2.718\ldots$. This is often called 'the' exponential function.

2. Logarithm functions

If we sketch graphs of logarithms to different bases we see there is a relationship between them

What is the effect of varying a? We can see by looking at sketches of a few graphs of similar functions. For the special case where $a = \mathrm{e}$, we often write $\ln x$ instead of $\log_{\mathrm{e}} x$.



We notice that the graphs all pass through the point (1,0) This is by definition of a logarithm. The power that any base must be raised to in order to achieve 1 is the power of zero.

So the log to the base of anything of 1 is zero

$\log_a 0 = 1$ because $a^0 = 1$

Natural Logarithms

Although Natural Logarithms do not play a part in the C2 course I will mention them here.

There is an <u>important irrational number denoted by e</u>. It is approximately equal to 2.71828……The reason for its importance is that if we DIFFERENTIATE (find the gradient of the tangent to any point) the equation of the curve $y=e^x$ then we get $\frac{dy}{dx} = e^x$. So e is the only number such that if we differentiate the power function we get the SAME function.

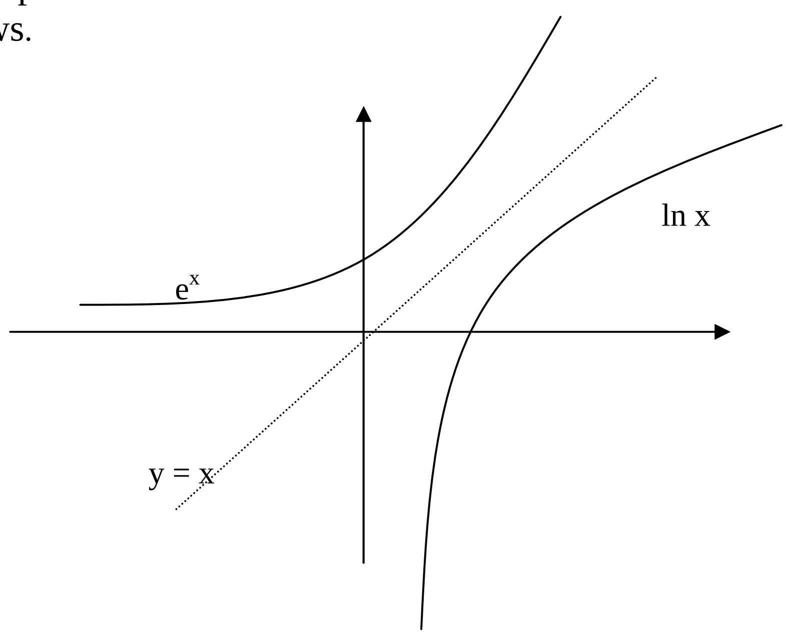
This value plays an important role in the modelling of population growth or decline, also it can be used to measure the rate of decay of radio-active material.

When e is used as the base for logarithms they are called <u>natural logarithms,</u> and to avoid any confusion they are referred to as ln x.

Generally if,

$$\boxed{\ln x = y \quad \Rightarrow \quad e^y = x}$$

It is important to realise that ln x and e are inverse functions, this can be shown graphically as follows.



NB.

ln 1 = 0 and e^0 = 1

(anything to the power of zero is equal to 1)

A LOGARITHM CAN NEVER BE A NEGATIVE NUMBER

Evaluating Logarithms with a calculator.

Logs can be evaluated mentally if the powers and bases are *convenient*,

eg. What is the value of $\log_4 64$?

$$4^3 = 64 \qquad \Rightarrow \quad \log_4 64 = 3$$

When they are more awkward we use the calculator,There are an INFINITE number of bases we could use but the Scientific calculator will have TWO bases on TWO different buttons. Neither button mentions the base but the base is *implied* in the following way:

log refers to the logarithm to the base of 10

ln refers to the logarithm to the base of e

LOOK FOR THESE BUTTONS ON YOUR CALCULATOR!

(remember log is in base 10.)

eg. $\log 2 = 0.301029995\ldots\ldots$

so $10^{0.301029995\ldots} = 2$

$\log 3 = 0.477121254\ldots\ldots$

so $10^{0.477121254\ldots} = 3$ (or almost!!!!)

C2 Examination content only uses logs to the base of 10

But all proofs in C2 will be to the base of a (any base)

REMEMBER THE LAWS OF LOGS WORK FOR ANY BASE

SUMMARY OF IMPORTANT POINTS

A Logarithm is a POWER

REMEMBER LOGS OF NEGATIVE NUMBERS DO NOT EXIST!!!

This is because there does not exist a number that you can use as a power that would result in a NEGATIVE number (if the BASE is positive)

Also

$$\log_a 0 = 1$$

because

$$a^0 = 1$$

no matter what the value of a

Because logs functions and exponential (power) functions are INVERSE Functions they will "cancel each other out"

$$\log_a a^x = x$$

$$a^{\log_a x} = x$$

VERY IMPORTANT IS HOW WE SWITHCH BETWEEN POWER STATEMENTS AND LOG STATEMENTS.

$$\textbf{If } \log_a b = c \iff a^c = b$$

Key Point

A function of the form $f(x) = \log_a x$ (where $a > 0$ and $a \neq 1$) is called a logarithm function.

The function $f(x) = \log_a x$ for $a > 1$ has a graph which is close to the negative $f(x)$-axis for $x < 1$ and increases slowly for positive x.

The function $f(x) = \log_a x$ for $0 < a < 1$ has a graph which is close to the positive $f(x)$-axis for $x < 1$ and decreases slowly for positive x.

For any value of a, the graph always passes through the point $(1, 0)$. The graph of $f(x) = \log_{1/a} x$ is a reflection, in the horizontal axis, of the graph of $f(x) = \log_a x$.

A particularly important logarithm function is $f(x) = \log_e x$, where $e = 2.718\ldots$. This is often called the natural logarithm function, and written $f(x) = \ln x$.