LOCI IN THE COMPLEX PLANE

Firstly we will look at Loci which should be learned and recognised.

WHAT IS A LOCUS? (What are loci?)

A LOCUS IS A PATH OF POSSIBLE POSITIONS OF A VARIABLE POINT, THAT OBEYS A GIVEN CONDITION. It can be given as a CARTESIAN EQUATION or it can be described in words.

EXAMPLE 1
What is the locus of the points which satisfy

$$|z| = r$$

The answer is that it will be a <u>circle centre at the origin with radius of r.</u>

This is because

$$z = x + iy$$
$$\text{so } |z| = \sqrt{(x^2 + y^2)} = r$$
$$x^2 + y^2 = r^2$$

using the definition of the MODULUS of a complex number.

$x^2 + y^2 = r^2$ is the Cartesian equation of a circle centre the origin with radius r. (You should know this from your C2 work on circles)

EXAMPLE 2
Find the locus of $|z - z_1| = r$
if $z_1 = u + iv$ is a known, fixed complex number and $z = x + iy$ is a variable.

ANSWER
The locus of $|z - z_1| = r$ is a circle centre z_1 and radius r.

This is because

$$|(x + iy) - (u + iv)| = r$$
$$|(x - u) - i(y - v)| = r$$
$$\sqrt{(x-u)^2 + (y-v)^2} = r$$
$$(x-u)^2 + (y-v)^2 = r^2$$

and by the definition of modulus

Which is the Cartesian equation of a circle centre (u,v) and radius r.
(Again with familiarity of C2 circles work)

2003 P5 Past Paper Question.
The complex number z is represented by the point P on the Argand diagram.

(a) Given that

$$|z-1-i| = |z-2|$$

find, in its simplest form, the Cartesian equation of the locus of P.

SOLUTION
Let z=x+iy

$$|(x+iy)-1-i| = |(x+iy)-2|$$
$$|(x-1)+i(y-1)| = |(x-2)+iy|$$
$$\sqrt{(x-1)^2+(y-1)^2} = \sqrt{(x-2)^2+y^2}$$

By the definition of the Modulus of a complex number.

$$x^2-2x+1+(y-1)^2 = (x-2)^2+y^2$$
$$x^2-2x+1+y^2-2y+1 = x^2-4x+4+y^2$$
$$-2x+1-2y+1 = -4x+4$$
$$2x-2y-2=0$$
$$y = x-1$$

So the Cartesian Equation is a straight line y=x-1

(b) Given that

$$|z-2| = 2|z+i|$$

show that the locus of P is a circle.

SOLUTION
Let z=x+iy

$$|(x+iy)-2| = 2|(x+iy)+i|$$
$$|(x-2)+iy| = 2|x+i(y+1)|$$
$$\sqrt{(x-2)^2+y^2} = 2\sqrt{x^2+(y+1)^2}$$
$$(x-2)^2+y^2 = 4(x^2+(y+1)^2)$$
$$x^2-4x+4+y^2 = 4(x^2+y^2+2y+1)$$
$$0 = 4x^2+4y^2+8y+4-(x^2-4x+4+y^2)$$
$$0 = 3x^2+3y^2+8y+4x$$
$$0 = 3(x^2+y^2+\frac{8}{3}y+\frac{4}{3}x)$$
$$0 = x^2+y^2+\frac{8}{3}y+\frac{4}{3}x$$

This is sufficient to justify that the locus is a circle as we are left with a cartesian equation of a circle.

We could find the centre and radius of the circle by completing the square in x and y but this was not required in his question.

JANUARY 2007 FP1

The complex number z is represented by the point P(x,y) in an Argand Diagram.

(a) Given that

$$|z-3| = |z+i|$$

find the Cartesian equation of the Locus of P.

(b) Find the two points lying on this locus for which $|z| = 4$

JUNE 2006 FP1

The complex numbers z, w are represented, respectively by the points P(x,y), Q(u,v) in Argand digrams and

$$w = z^2$$

P moves along the line y=x-1 .
Find the Cartesian equation of the locus of Q.

THIS QUESTION IS VERY MUCH LIKE JUNE 2006 FP1

The complex numbers z, w re represented by the points P(x,y), Q(u,v) in Argand digrams and

$$w = z^2$$

(a) Find expressions for u and v in terms of x and y.
Given that P move along the line x+y=1, find the Cartesian equation of the locus of Q.

1995 LEGACY PAPER

The complex numbers z and w are represented by the points P(x,y) and Q(u,v) respectively in Argand diagrams and

$$w = z^2$$

(a) show that

$$u = x^2 - y^2$$

and find an expression for v in terms of x and y.

(b) The point P moves along the curve with equation $2xy^2 = 1$

(i) Show that

$$v = \frac{1}{y}$$

(ii) find the locus of Q, giving your answer in the form u=f(v)

Another example (not past paper)

The compex number z is represented by the point on an Argand diagram.

Given that

$$\left|\frac{z-1}{z+1}\right| = 2$$

show that the locus of P is a circle.

State its radius and the coordinates of the centre.

2005 FP1 NEW SPECIFICATION

The complex numbers z and w are represented, respectively, by the points P(x,y) and Q(u,v) respectively in Argand diagrams and

$$w = \frac{1}{z}$$

(a) show that

$$x = \frac{u}{u^2 + v^2}$$

and find an expression for y in terms of u and v.

(b) the point P moves along the circle $x^2 + y^2 = 2$. Find the equation of the locus of Q in the (u,v) plane.

JANUARY 2006 FP1

The complex numbers z and w are represented, respectively, by the points P(x,y) and Q(u,v) respectively in Argand diagrams and

$$w = \frac{z+3}{z+1}$$

The point moves around the circle with equation $|z| = 1$.

Find the Cartesian equation of the locus of Q. Identify this locus

NOT A PAST PAPER

The locus L in the Argand diagram has equation

$$|z-2-4i|=|z-4-6i|$$

Find the cartesian equation of L showing it to be a straight line

NOT PAST PAPER

The complex numbers z=x+iy and w=u+iv are represented in the Argand diagram by the points P and Q respectively. Given that

$$w = (z+2)^2 + 5$$

find u and v in terms of x and y.

If P moves along the line x=0, find the equation of the locus of Q in the form u=f(v)