INVERSE HYPERBOLIC FUNCTIONS

We will recall that the function

$$\sinh x = \frac{e^x - e^{-x}}{2}$$ is a 1 to 1 mapping.ie one value of x maps onto one value of sinh*x*.

This implies that the inverse function exists.

$$y = \sinh x \Leftrightarrow x = \sinh^{-1} y$$

The function f(x)=sinh*x* clearly involves the exponential function.

We will now find the inverse function in its logarithmic form.
LET

$$y = \sinh^{-1} x$$
$$x = \sinh y$$

THEN

$$x = \frac{e^y - e^{-y}}{2}$$ (BY DEFINITION)

SOME ALGEBRAIC MANIPULATION GIVES

$$2x = e^y - e^{-y}$$

MULTIPLYING BOTH SIDES BY e^y

$$2xe^y = e^{2y} - 1$$

RE-ARRANGING TO CREATE A QUADRATIC IN e^y

$$0 = e^{2y} - 2xe^y - 1$$

APPLYING THE QUADRATIC FORMULA

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$$e^y = \frac{2x \pm 2\sqrt{x^2+1}}{2}$$

$$e^y = x \pm \sqrt{x^2+1}$$

SINCE

$$e^y > 0$$

WE MUST ONLY CONSIDER THE ADDITION.

$$e^y = x + \sqrt{x^2+1}$$

TAKING LOGARITHMS OF BOTH SIDES

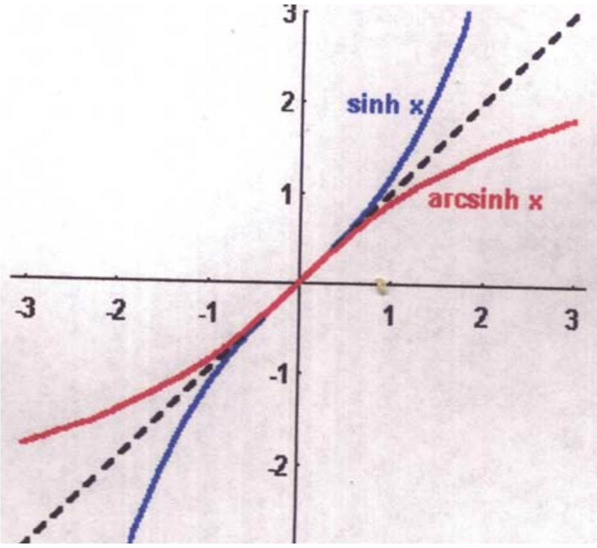
$$y = \ln\left\{x + \sqrt{x^2+1}\right\}$$

IN CONCLUSION

$$\sinh^{-1} x = \ln\left\{x + \sqrt{x^2+1}\right\}$$

THIS IS CALLED THE LOGARITHMIC FORM OF THE INVERSE FUNCTION.

The graph of the inverse function is of course a reflection in the line y=x.



THE INVERSE OF COSH*X*

The function

$$f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$$

Is of course not a one to one function. (observe the graph of the function earlier).

In order to have an inverse the DOMAIN of the function is restricted.

$$f(x) = \cosh x = \frac{e^x + e^{-x}}{2}, x \geq 0$$

This is a one to one mapping with RANGE

$[1, \infty]$.

We find the inverse function in logarithmic form in a similar way.

$$y = \cosh^{-1} x$$

$$x = \cosh y$$

provided $y \geq 0$

$$x = \cosh y = \frac{e^y + e^{-y}}{2}$$

$$2x = e^y + e^{-y}$$

$$0 = e^y - 2x + e^{-y}$$

$$0 = e^{2y} - 2xe^y + 1$$

USING QUADRATIC FORMULA

$$e^y = \frac{-2x \pm \sqrt{4x^2 - 4}}{2}$$

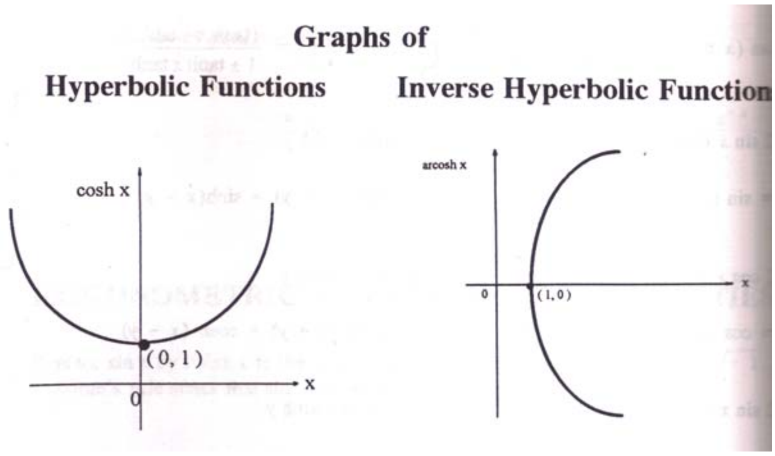
$$e^{y} = x \pm \sqrt{x^{2} - 1}$$

TAKING LOGS

$$y = \ln\left\{x \pm \sqrt{x^{2} - 1}\right\}$$

$y = \ln\left\{x + \sqrt{x^{2} - 1}\right\}$ is one possible expression

APPROPRIATE IF THE DOMAIN OF THE ORIGINAL FUNCTION HAD BEEN RESTRICTED AS EXPLAINED ABOVE.



HOWEVER CONSIDERING THE WHOLE FUNCTION (UNRESTRICTED) FOR EACH VALUE OF X THERE ARE TWO EQUAL AND OPPOSITE VALUES OF Y.

This implies

$$\cosh^{-1} x = \pm \ln\left\{x + \sqrt{x^{2} - 1}\right\}$$

To add a more sound mathematical argument to this we can use some algebraic manipulation and laws of logs to the alternative solution.

Consider

$$x - \sqrt{x^{2} - 1} \equiv \frac{(x - \sqrt{x^{2} - 1})(x + \sqrt{x^{2} - 1})}{(x + \sqrt{x^{2} - 1})}$$

THIS IS NOTHING BUT A USEFUL TRICK!

The numerator becomes 1 (check it and see!)

So

$$x-\sqrt{x^2-1} \equiv \frac{1}{(x+\sqrt{x^2-1})} \equiv (x+\sqrt{x^2-1})^{-1}$$

Now returning to out alternative solution

$$y = \cosh^{-1} x = \pm \ln\left\{x+\sqrt{x^2-1}\right\}$$

Covers all possible solutions depending on the restriction of the domain

TASK: Show that the logarithmic form of the hyperbolic tan is

$$y = \tanh^{-1} x = \tfrac{1}{2}\ln\frac{1+x}{1-x} \text{ for } x\angle 1$$

HINT

Start as before by rearranging the inverse statement and then use

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

THE GOOD NEWS!
THE FORMULAE FOR THE LOGARITHMIC FORMS OF INVERSE HYPERBOLIC FUNCTIONS ARE IN THE WJEC FORMULA BOOK!