HYPERBOLIC FUNCTIONS.

The **HYPERBOLIC FUNCTIONS** enjoy properties similar to the trigonometric or **CIRCULAR FUNCTIONS**.

The **CIRCULAR FUNCTIONS** are so called because x=cos(t), y=sin(t) are the **PARAMETRIC EQUATIONS of CIRCLE** radius 1.
$x^2 + y^2 = 1$.
Hence the name, **CIRCULAR FUNCTIONS**.

The **HYPERBOLIC FUNCTIONS** are so called because x=cosh(t), y=sinh(t) are the **PARAMETRIC EQUATIONS** of the **STANDARD HYPERBOLA** $x^2 - y^2 = 1$

We define the **HYPERBOLIC FUNCTIONS** as follows:

$$\boxed{\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2} \\ \cosh x &= \frac{e^x + e^{-x}}{2} \end{aligned}}$$

Cosh is pronounced "KOSH"
Sinh is pronounced to rhyme with "GRINCH"

These two relations when added and subtracted give:

$$e^x = \cosh x + \sinh x \quad \text{and} \quad e^{-x} = \cosh x - \sinh x$$

NOTE ALSO:

$$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^x}{2} = \cosh x$$

$$\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = \frac{e^{-x} - e^x}{2} = \frac{-(e^x - e^{-x})}{2} = -\sinh x$$

The graphs of these two HYPERBOLIC FUNCTIONS are shown :.

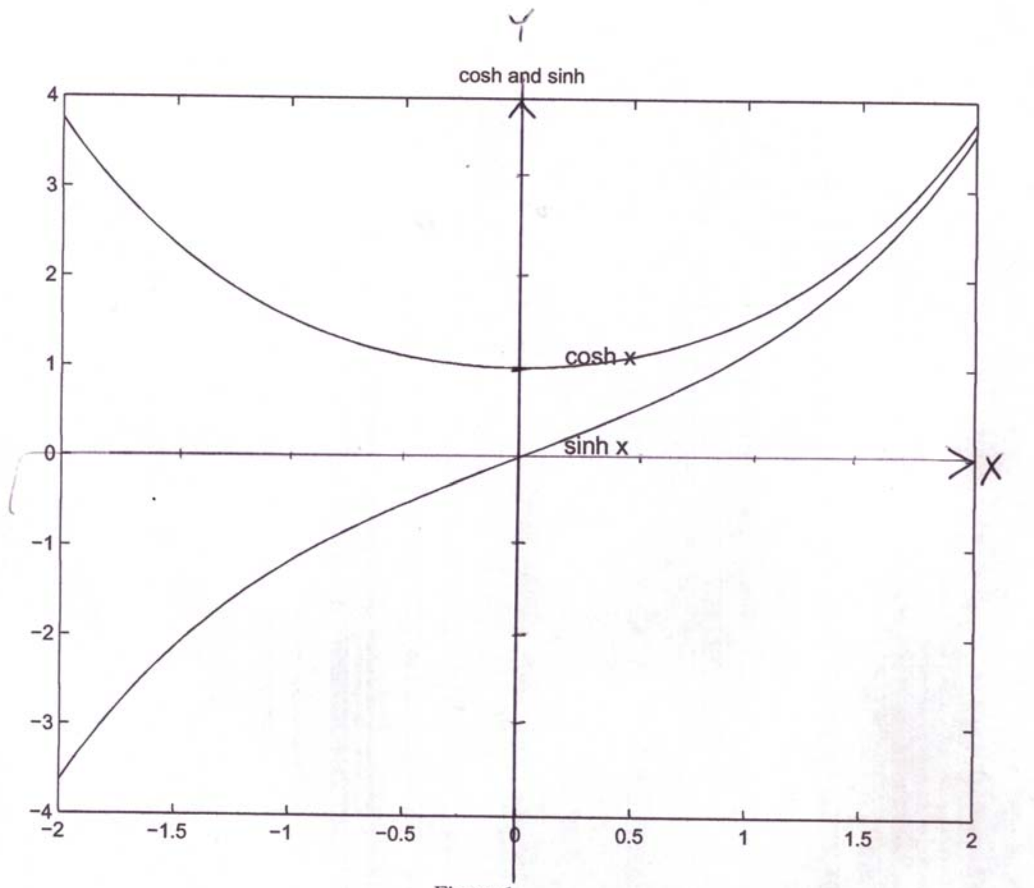

Figure 1

It should be noted:

- cosh x is an **EVEN FUNCTION**

An even function is such that
$f(-x) = f(x)$ for all values of x in its domain.

The graph is symmetric about the vertical axis
Domain is $(-\infty, \infty)$ RANGE is $[1, \infty)$

- sinh x is an **ODD FUNCTION**

An odd function is such that
$f(-x) = -f(x)$ for all values of x in its domain

The graph looks the same when it is rotated through half a revolution about O.
Domain is $(-\infty, \infty)$ Range is $(-\infty, \infty)$
As $x \to \infty$ both sinh*x* and cosh $x \to \infty$ with sinhx less than coshx.

FURTHER HYPERBOLIC FUNCTIONS:

$$\tanh x = \frac{\sinh x}{\cosh x}$$

Alternatively we could define tanh x directly in terms of the exponential functions;

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\tanh(-x) = \frac{\sinh(-x)}{\cosh(-x)} = \frac{-\sinh x}{\cosh x} = -\tanh x$$

So tanh x is also an **ODD FUNCTION** as
$f(-x) = -f(x)$ for all values of x in its domain

We may also define:

$$\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\text{cosech}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

in the same way as the trigonometric functions.
Clearly there must be other similarities between the trigonometric and Hyperbolic functions.

THE EQUIVALENT OF PYTHAGORAS' THEOREM.

$$\cosh^2 x = \left[\frac{e^x + e^{-x}}{2}\right]^2 = \frac{e^{2x} + 2 + e^{-2x}}{4} = \tfrac{1}{4}e^{2x} + \tfrac{1}{2} + \tfrac{1}{4}e^{-2x}$$

$$\sinh^2 x = \left[\frac{e^x - e^{-x}}{2}\right]^2 = \frac{e^{2x} - 2 + e^{-2x}}{4} = \tfrac{1}{4}e^{2x} - \tfrac{1}{2} + \tfrac{1}{4}e^{-2x}$$

SUBTRACTING GIVES:

$$\boxed{\cosh^2 x - \sinh^2 x = 1}$$

NOTE THAT **ADDING GIVES:**

$$\cosh^2 x = \left[\frac{e^x + e^{-x}}{2}\right]^2 = \frac{e^{2x} + 2 + e^{-2x}}{4} = \tfrac{1}{4}e^{2x} + \tfrac{1}{2} + \tfrac{1}{4}e^{-2x}$$

$$\sinh^2 x = \left[\frac{e^x - e^{-x}}{2}\right]^2 = \frac{e^{2x} - 2 + e^{-2x}}{4} = \tfrac{1}{4}e^{2x} - \tfrac{1}{2} + \tfrac{1}{4}e^{-2x}$$

$$\cosh^2 x + \sinh^2 x = \tfrac{1}{2}(e^{2x} + e^{-2x}) = \cosh 2x$$

So there is again a similarity with ’he trigonometric Double angle identities.

TASK1

Prove that $\sinh 2x \equiv 2\sinh x \cosh x$

Show that $\cosh(x+y) \equiv \cosh x \cosh y + \sinh x \sinh y$

STARTING WITH THE RIGHT HAND SIDE IN EXPONENTIAL FORM

$$\cosh x \cosh y + \sinh x \sinh y \equiv \frac{e^x + e^{-x}}{2}\frac{e^y + e^{-y}}{2} + \frac{e^x - e^{-x}}{2}\frac{e^y - e^{-y}}{2}$$

EXPANDING GIVES

$$\tfrac{1}{4}((e^{x+y} + e^{x-y} + e^{y-x} + e^{-x-y}) + (e^{x+y} - e^{x-y} - e^{y-x} + e^{-x-y}))$$

$$= \tfrac{1}{4}.(2e^{x+y} + 2e^{-(x+y)})$$

$$= \cosh(x+y)$$

QED.

TASK 2:

SHOW THAT $\sinh(x+y) \equiv \sinh x \cosh y + \cosh x \sinh y$

STARTING WITH $\cosh^2 x - \sinh^2 x = 1$

We can derive other identities similar to the Trig Pythagorean identities.

If you can divide by $\cosh^2 x$ you obtain

$$1 - \frac{\sinh^2 x}{\cosh^2 x} \equiv \frac{1}{\cosh^2 x}$$

that is

$$1 - \tanh^2 x \equiv \operatorname{sech}^2 x$$

TASK 3:
SHOW THAT

$$\coth^2 x - 1 \equiv \operatorname{cosech}^2 x \quad \text{for } x \neq 0$$

OSBORNE'S RULE

There is a clear similarity between trigonometric identities and hyperbolic identities. Many are of the same form often with signs changed,(but not always).

OSBORNE'S RULE gives a simple way to remember when to make a sign change when moving from a trigonometric identity to a hyperbolic.

The rule is to replace each trigonometric function by the corresponding hyperbolic and change the sign of every product (or implied product) of two sines. Note that OSBORNE'S RULE is used to find identities but not to prove them. It comes about because sin*x* is replaced by *i*sinh*x* where $i = \sqrt{-1}$. So $\sin^2 x$ is replaced by $-\sinh^2 x$.

A table of corresponding Trig and Hyperbolic identities follows.

TRIGONOMETRIC	HYPERBOLIC
$\sin^2 x + \cos^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
$1 + \tan^2 x = \sec^2 x$	$1 - \tanh^2 x = \text{sech}^2 x$
$1 + \cot^2 x = \text{cosec}^2 x$	$1 - \coth^2 x = -\text{cosech}^2 x$
$\cos 2x = 2\cos^2 x - 1$	$\cosh 2x = 2\cosh^2 x - 1$
$\cos 2x = 1 - 2\sin^2 x$	$\cosh 2x = 1 + 2\sinh^2 x$
$\cos x = 2\cos^2 \frac{x}{2} - 1$	$\cosh x = 2\cosh^2 \frac{x}{2} - 1$
$\cos x = 1 - 2\sin^2 \frac{x}{2}$	$\cosh x = 1 + 2\sinh^2 \frac{x}{2}$
$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$	$\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$
$\sin(x \pm y)$	$\sinh(x \pm y)$
$= \sin x \cos y \pm \sin y \cos x$	$= \sinh x \cosh y \pm \sinh y \cosh x$
$\cos(x \pm y)$	$\cosh(x \pm y)$
$= \cos x \cos y \mp \sin x \sin y$	$= \cosh x \cosh y \pm \sinh x \sinh y$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$\tanh(x \pm y) = \frac{(\tanh x \pm \tanh y)}{1 \pm \tanh x \tanh y}$$

$2 \sin x \cos y$

$= \sin(x + y) + \sin(x - y)$

$2 \sinh x \cosh y$

$= \sinh(x + y) + \sinh(x - y)$

$2 \cos x \cos y$

$= \cos(x + y) + \cos(x - y)$

$2 \cosh x \cosh y$

$= \cosh(x + y) + \cosh(x - y)$

$2 \sin x \sin y$

$= \cos(x - y) - \cos(x + y)$

$- 2 \sinh x \sinh y$

$= \cosh(x - y) - \cosh(x + y)$

$\sin x + \sin y$

$$= 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}$$

$\sinh x + \sinh y$

$$= 2 \sinh \frac{x + y}{2} \cosh \frac{x - y}{2}$$

$\sin x - \sin y$

$$= 2 \cos \frac{x + y}{2} \sin \frac{x - y}{2}$$

$\sinh x - \sinh y$

$$= 2 \cosh \frac{x + y}{2} \sinh \frac{x - y}{2}$$

$\cos x + \cos y$

$$= 2 \cos \frac{x + y}{2} \cos \frac{x - y}{2}$$

$\cosh x + \cosh y$

$$= 2 \cosh \frac{x + y}{2} \cosh \frac{x - y}{2}$$

$\cos x - \cos y$

$$= - 2 \sin \frac{x + y}{2} \sin \frac{x - y}{2}$$

$\cosh x - \cosh y$

$$= 2 \sinh \frac{x + y}{2} \sinh \frac{x - y}{2}.$$