Co-ordinate Geometry
THE EQUATION OF STRAIGHT LINES

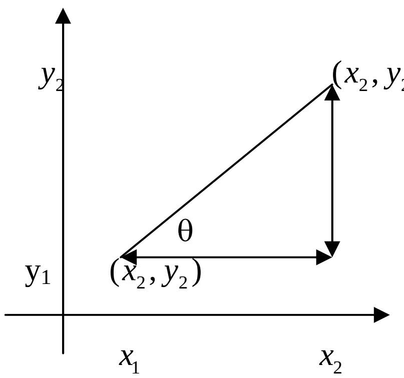
This section refers to the properties of straight lines and curves using rules found by the use of cartesian co-ordinates.

The Gradient of a Line.

As we have met the gradient of a line at GCSE but we will now formally define it as the "rate of change of y with respect to x".
It is visually the steepness or slope of the line and is often denoted by the letter "m". (Due to the idea being developed by a French man named Rene Decartes. The French for to climb is MONTER)

Gradient (m) = $\frac{\text{change in y}}{\text{Change in x}}$

Consider

 Now consider the general situation.	Therefore gradient = $\frac{y_2 - y_1}{x_2 - x_1}$ OR **Gradient = Tan θ**

OR

GRADIENT (m) = $\frac{y_2 - y_1}{x_2 - x_1}$

Where two co-ordinate pairs are known.

Example 1.
Find the gradient of the line adjoining A = (2, 4) and B = (5, 9).

Decide which point will represent (x_2, y_2)
and which will represent (x_1, y_1)
They must work as a pair and must not be mixed up.

$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{4 - 9}{2 - 5}$$

$$m = \frac{-5}{-3}$$

m = 5/3.

N.B. The same gradient will result from the calculation if the co-ordinates were chosen the other way around. Try it and check!!!

Exercise.

Find the gradient of the line adjoining the following pairs of points.

	A	B
1.	(-1, 4)	(-3, -8)
2.	(-2, -5)	(3, -5)
3.	(-2, -5)	(-3, -3)
4.	(1, 4)	(2, 9)
5.	(2, 5)	(6, 6)
6.	(9, 4)	(8, 7)
7.	(3, 8)	(-1, 2)
8.	(4, 3)	(-2, -3)
9.	(-2, 5)	(3, 1)
10.	(1, -4)	(6, 2)

Exercise solutions.
6, 0 , -2, 5, 1/4, -3, 3/2 , 1 , -4/5 , 6/5

The Equation of a Straight Line.

Any straight line can be represented by the general equation

$ax + by + c = 0$ where a, b, and c are constants.

In this form it is difficult to see the gradient and intercept value. These are important as they allow us to sketch the function quickly.

Another way of representing a straight line is by the general equation

$$y = mx + c,$$

where m represents gradient and c represents the intercept of the function.

We already know that $m = \frac{y_2 - y_1}{x_2 - x_1}$ denoting (x_2, y_2) by just (x,y) and

By rearranging this gradient formula we get a third general formula for straight line function. The IMPORTANT VERSION of the EQUATION OF A STRAIGHT LINE is obtained.

$$y - y_1 = m(x - x_1)$$

Where (x_1, y_1) represents any point on the line

Example 1.

Find the equation of the line passing through the points (2, 3) and (3, 7).

Method.

1. Find the gradient.

$$m = \frac{y - y_1}{x - x_1}$$

$$m = \frac{7 - 3}{3 - 2}$$

$$m = 4$$

2. Using the general equation find the required straight line solution.

(using point (2, 3))

$$y - y_1 = m(x - x_1)$$

$$y - 3 = 4(x - 2)$$

$$y - 3 = 4x - 8$$

OR

$$y = 4x - 5$$

Example 2.

Find the equation of the line with gradient –1/3 which passes through point (1, -2)

Since m = -1/3

$$y - y_1 = m(x - x_1)$$

$$y - (-2) = \frac{-1}{3}(x - 1)$$

$$3(y + 2) = -x + 1$$

$$3y + 6 = -x + 1$$

$$3y = -x - 5$$

OR

$$y = \frac{-x - 5}{3}$$

EXAMPLE 3

ABCD is a rectangle. *A* has co-ordinates (4, 6), *B*, (–2, 4) and *C*, (–1, 1).

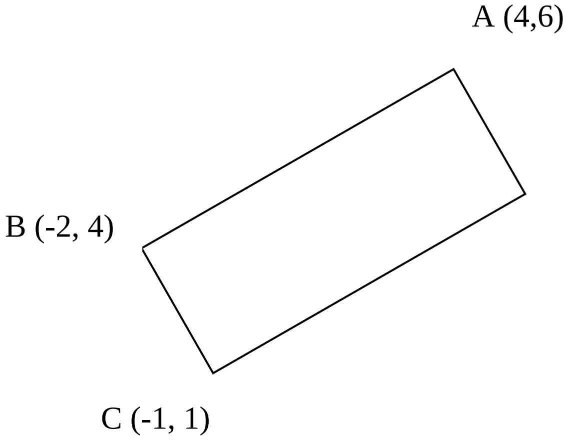
(a) Find the equation of *AD*.

(b) Find the equation of *CD*.

There seems to be a problem in that we do not know the coordinates of D!!

However we do know that we have a rectangle and so we have PARALLEL LINES.

IT IS A SIMPLE FACT THAT PARALLEL LINES HAVE EQUAL GRADIENTS.



It is unimportant where the x and y axes are!

We can see from this that BC is PARALLEL to AD
And AB is PARALLEL to DC

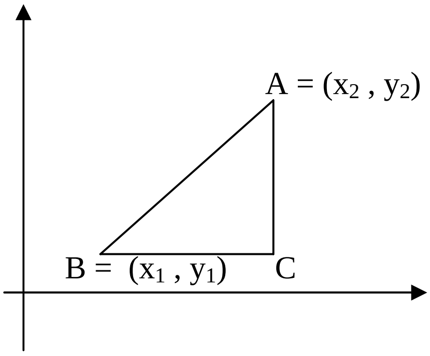
We will use this to answer the problem

ANSWER:

TO FIND THE EQUATION OF AD

TO FIND THE EQUATION OF CD

The Distance Between Two Points.



Let A = (x_2, y_2), B = (x_1 , y_1) and C is a point to create a right-angled triangle.

We can see that by using Pythagoras Theorem that;

$$AB^2 = BC^2 + AC^2$$

But, length BC is (x_2 – x_1) and length AC is (y_2 – y_1)

Therefore

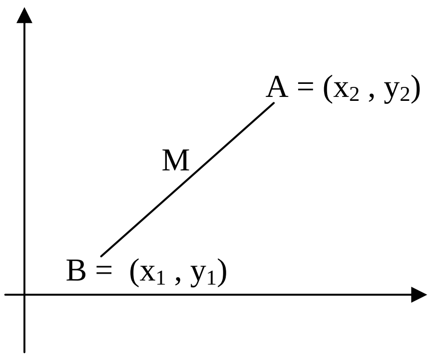
$$AB^2 = (\mathbf{x_2 - x_1})^2 + (\mathbf{y_2 - y_1})^2$$

$$\mathbf{AB} = \sqrt{(\mathbf{x_2 - x_1})^2 + (\mathbf{y_2 - y_1})^2}$$

The distance between two (x_1, y_1) and (x, y) points is found by using

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The Mid-Point of a Line.



Therefore the co-ordinate of the mid-point of AB is given by

$$\left\{ \frac{x_2 + x_1}{2} \ , \ \frac{y_2 + y_1}{2} \right\}$$

Example.

a) Find the distance from the point A = (-7, 1) and B = (7, 6).

b) Write down the co-ordinates of M the mid point of AB.

$$AB = \sqrt{(x - x_1)^2 + (y - y_1)^2}$$

$$AB = \sqrt{(7 - (-7))^2 + (6 - 1)^2}$$

$$AB = \sqrt{221}$$

AB = 14.87 units.

$$M = \frac{x_2 + x_1}{2} \ , \ \frac{y_2 + y_1}{2}$$

M = (0, 3.5)

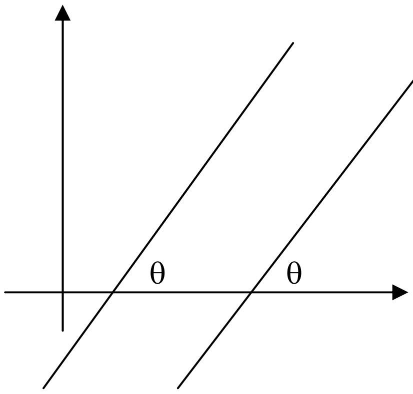
Exercise

a) Find the equation of the straight line passing through the following points A and B.
b) Find the distance AB for questions 1,3 and 5
c) Find M the mid point of AB
.

	A		B	
1.	(2, 4)		(3, 7)	
2.	(-1, 2)		(1, -4)	
3.	(-2, -3)		(7, 2)	
4.	(3, -1)		(-2, 3)	
5.	(2, 0)		(-1, 3)	

Parallel and Perpendicular Lines.

Parallel Lines. As we have already seen



Any two lines that are parallel to each other have equal inclines to the x axis.

If they have equal inclines then the rate of change of y with respect to x must be equal

For lines to be parallel they must have equal gradients.

Example 1.
Given A = (-2, 0), B = (0, 1), C = (3, -4) and D = (5, -3).
Show that AB is parallel to CD.

Gradient AB

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$m = \frac{1 - 0}{0 - (-2)}$

$m = 1/2$

Gradient CD

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$m = \frac{-4 - (-3)}{3 - 5}$

$m = -1/ -2$

$m = 1/2$

Since the gradients are equal, we can conclude that the lines AB and CD are parallel.

Example 2.

Show that the points A =(-2, 5), B = (7, 2), C = (3, -2) and D = (-6,1) are vertices of a parallelogram.

A parallelogram has two pairs of parallel sides by definition.

That is AB must be parallel to CD and BC must be parallel to AD.

Gradient AB

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{5 - 2}{-2 - 7}$$

$$m = -1/3$$

Gradient BC

$$m = \frac{-2 - 2}{3 - 7}$$

$$m = 1$$

Gradient CD

$$m = \frac{-2 - 1}{3 - (-6)}$$

$$m = -1/3$$

same as AB

Gradient AD

$$m = \frac{5 - 1}{-2 - (-6)}$$

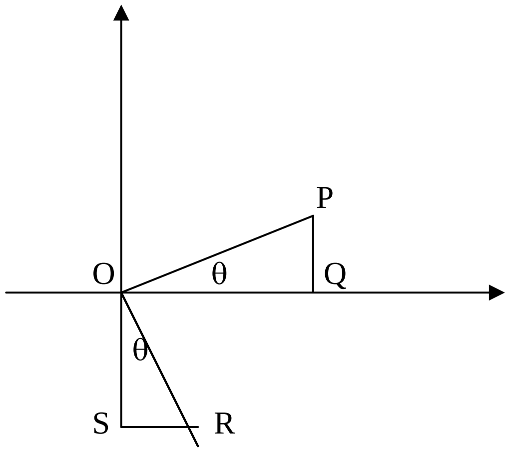
$$m = 1$$

same as BC

Therefore a parallelogram!

Perpendicular Lines.

The Proof is not necessary but have a look anyway.
Consider the diagram below.



Lines OP and OR are perpendicular with gradients m_1 and m_2 respectively.

If OP makes an angle of θ with the x axis then OR makes an angle of θ with the y axis.

We can see that triangles OPQ and OSR are similar.

Therefore $\frac{PQ}{OQ} = \frac{SR}{OS}$

But the gradient of OP = $\frac{PQ}{OQ}$

And the gradient of OR = $-\frac{OS}{RS}$

But the gradient of OP = m_1, and the gradient of OR = m_2.

$\Rightarrow \quad m_1 = \frac{-1}{m_2}$

OR $\quad m_1 . m_2 = -1$

For lines to be perpendicular the products of their gradients must be -1.

It follows that if the gradient of one line is m then the gradient of any perpendicular line is $\frac{-1}{m}$.

Example 1.

Show that the lines 4y – 3x – 18 = 0 and 3y + 4x – 1 = 0 are perpendicular.

For lines to be perpendicular the products of their gradients must be -1.
We find the gradient by re arranging the equation into the form y=mx+c

Line 1; $\quad y = \frac{3x + 18}{4} \quad$ gradient = 3/4

Line 2; $\quad y = \frac{-4x + 1}{3} \quad$ gradient = -4/3

3/4 x –4/3 = -1 therefore lines are perpendicular.

Exercise.

1. Show that the lines $y - 2x - 1 = 0$ and $6y + 3x - 5 = 0$ are perpendicular.

2. Test whether the lines $5y - 7x - 6 = 0$ and $6y + 5x + 3 = 0$ are perpendicular.

3. Find the equation of a straight line which is perpendicular to the line

4. $y = 2x + 1$ which passes through the point (3, -2).

5. Find the equation of a straight line which is parallel to the line
$y = -5/4x + 2$ which passes through the point (4, 3).

6. Find the equation of a straight line passing through (5, 2) which is perpendicular to the line $2x + 3y - 4 = 0$.

Equation Revision Sheet.

1. For the following equations, state
a) the gradient of the line.
b) the gradient of the perpendicular to the line.

$$4y = x + 5$$
$$2y = 4/5x + 3$$
$$3x + 5y = 6$$
$$4 - 5y = 4x$$
$$7y - 5/3x = 8$$

2. Find the equation of the line passing through (4, 3) which is parallel to $2x + 3y = 7$.

3. Find the equation of a line which passes through (4, 3) which is perpendicular to the line $2x + 3y = 7$.

4. Find the equation of a line which passes through (1, 4) which is perpendicular to the line $3y + 4x = 7$.

5. Find the equation of a line which passes through (2, 2) which is parallel to the line $5x - 2y = 4$.

6. Find the equation of a line which passes through (1, 6) which is perpendicular to the line $3x = 2 - 7y$.

7. ABCD is a parallelogram, where A = (-4, 1), B = (5, -2) and C = (3, 3). Calculate;
a) The gradient of AB and BC.
b) The equations of the lines AD and CD.
c) The co-ordinates of D.
d) The co-ordinates of the points of intersection of AC and BD.