Differentiation of Inverse Hyperbolic Functions

1. $y = \sinh^{-1} x$

$$y = \sinh^{-1} x$$
$$x = \sinh y$$
$$\frac{dx}{dy} = \cosh y$$
$$\frac{dy}{dx} = \frac{1}{\cosh y}$$

but $\cosh^2 y - \sinh^2 y \equiv 1$

so $\cosh y = \sqrt{1 + \sinh^2 y}$

giving
$$y = \sinh^{-1} x$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$$

NOTE that the positive square root is taken since $y = \sinh^{-1} x$ is a monotonic increasing function ie the gradient is always positive and so $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+x^2}}$ is always greater than zero.

This result has an important consequence for integration:

$$y = \sinh^{-1} x \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} \qquad \int \frac{1}{\sqrt{1+x^2}}dx = \sinh^{-1} x$$

This is a similar result to the inverse trigonometric functions:

$$y = \sin^{-1} x \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x$$

In a more general case:

$$y = \sinh^{-1}\left(\frac{x}{a}\right) \qquad \frac{dy}{dx} = \frac{1}{\sqrt{a^2+x^2}} \qquad \int \frac{1}{\sqrt{a^2+x^2}}dx = \sinh^{-1}\frac{x}{a}$$

Differentiation of Inverse Hyperbolic Functions

2. $y = \cosh^{-1} x$

$$y = \cosh^{-1} x$$
$$x = \cosh y$$
$$\frac{dx}{dy} = \sinh y$$
$$\frac{dy}{dx} = \frac{1}{\sinh y}$$

but $\cosh^2 y - \sinh^2 y \equiv 1$

so $\sinh y = \sqrt{\cosh^2 y - 1}$

giving
$$y = \cosh^{-1} x$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}$$

This result also has an important consequence for integration:

$$y = \cosh^{-1} x \qquad \frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}} \qquad \int \frac{1}{\sqrt{x^2 - 1}} dx = \cosh^{-1} x$$

This is a similar result to the inverse trigonometric functions but here we seldom use the inverse cos equivalent as it is the same result as for the inverse sin derivative with the exception only of a negative:

$$y = \cos^{-1} x \qquad \frac{dy}{dx} = \frac{-1}{\sqrt{1 + x^2}} \qquad \int \frac{-1}{\sqrt{1 + x^2}} dx = \cos^{-1} x = -\sin^{-1} x$$

In a more general case:

$$y = \cosh^{-1}\left(\frac{x}{a}\right) \qquad \frac{dy}{dx} = \frac{1}{\sqrt{x^2 - a^2}} \qquad \int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1} \frac{x}{a}$$

The proof of this can be found on page 203 of Further Pure Maths B and M Gaulter.

NOW try the same for $y = \tanh^{-1} x$

$y = \tanh^{-1} x$

$x =$

$\frac{dx}{dy} =$

$\frac{dy}{dx} =$

Remembering the identity

$\sec h^2 y \equiv 1 - \tanh^2 y$

$\frac{dy}{dx} =$

$y = \tanh^{-1}\left(\frac{x}{a}\right)$

$\frac{dy}{dx} = \frac{1}{a^2 - x^2}$

$$\int \frac{1}{a^2 - x^2} dx = \tanh^{-1} \frac{x}{a}$$

The proof of this can be found on page 203 of Further Pure Maths B and M Gaulter.

However we notice that $\int \frac{1}{a^2 - x^2} dx = \int \frac{1}{(a-x)(a+x)} dx$ which could easily be done by splitting the function into its partial fractions. This gives the logarithmic form of the answer which is quoted in the formula booklet and more commonly used.

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \int \left(\frac{1}{a+x} + \frac{1}{a-x}\right) dx$$
$$= \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + c$$

All standard integrals are given in the formula booklet: