A Couple of Proofs of De Moivre’s Theorem

& My Favourite Piece of Maths

©2008 Kai Reakes

De Moivre’s Theorem says:

\[(\cos(\theta) + i \sin(\theta))^n = \cos n\theta + i \sin n\theta\]

where, we remember, \(i^2 = -1\). I’ll prove this in two ways. Firstly by induction (which is the way the syllabus says to prove it), and secondly using the exponential form of a complex number (which I’ll also discuss). At the end I will show what I think is one of the most beautiful results in mathematics.

**Proof of De Moivre’s Theorem by Induction**

We just show De Moivre for positive integer values of \(n\). Remember we need our starting step for induction proofs. For \(n = 1\) we have:

\[(\cos(\theta) + i \sin(\theta))^1 = \cos 1\theta + i \sin 1\theta.\]

So De Moivre holds for \(n = 1\). Assume now it’s true for \(n = k\). I.e.:

\[(\cos(\theta) + i \sin(\theta))^k = \cos k\theta + i \sin k\theta.\] (1)

We want to show it is also true for \(n = k + 1\), and then we’ll be done. Consider then:

\[(\cos(\theta) + i \sin(\theta))^{k+1} = (\cos(\theta) + i \sin(\theta))(\cos(\theta) + i \sin(\theta))^k.\]

We now use equation (1), and sub in for \((\cos(\theta) + i \sin(\theta))^k\) on the right hand side. Then we get

\[(\cos(\theta) + i \sin(\theta))^{k+1} = (\cos(\theta) + i \sin(\theta))(\cos k\theta + i \sin k\theta)).\]

\[= \cos(\theta) \cos(k\theta) + i^2 \sin(\theta) \sin(k\theta) + i \sin(\theta) \cos(k\theta) + i \sin(k\theta) \cos(\theta)\]

\[= \cos(\theta) \cos(k\theta) - \sin(\theta) \sin(k\theta) + i(\sin(\theta) \cos(k\theta) + \sin(k\theta) \cos(\theta))\]

after expanding the brackets. Now, as you all know (without having to look it up in your formula booklet, of course), \(\sin (a + b) = \sin a \cos b + \sin b \cos a\) and \(\cos (a + b) = \cos a \cos b - \sin a \sin b\). So we have

\[(\cos(\theta) + i \sin(\theta))^{k+1} = \cos (k + 1)\theta + i \sin (k + 1)\theta.\]

But this is just what we need, right? So by induction we’ve shown De Moivre to be true for all positive integers \(n\).
Exponential Form of a Complex Number

So we know that we can write any complex number \( z = a + ib \), where \( a \) and \( b \) are real numbers. From drawing a triangle in the argand diagram, we know it can also be written as \( z = r(\cos \theta + i \sin \theta) \), where \( r \geq 0 \) and \( 0 \leq \theta \leq 2\pi \). In fact, \( r \) is the length of the line joining \( z \) with the origin (\( r \) is called the modulus), so by Pythagoras’ Theorem, \( r = \sqrt{a^2 + b^2} \). Also, \( \theta \) is the angle this line makes with the \( x \)-axis (going anticlockwise from the \( x \)-axis, by convention) (\( \theta \) is called the argument). So, by trigonometry, \( \theta = \tan^{-1}(b/a) \). In fact, with the same \( r \) and \( \theta \), we can write \( z = re^{i\theta} \), (and I’ll leave you to try to make any sense of what it means to have a power that is a complex number). So let’s try to show we can write \( z \) in this way.

Firstly, recall the Taylor (or Maclaurin) series for \( \sin x \), \( \cos x \) and \( e^x \):

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots \\
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots \\
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots
\]

These are also in the formula booklet. Let’s plug in \( \theta \) for \( x \) in the expressions for \( \sin x \) and \( \cos x \). And lets multiply the expression for \( \sin x \) by \( i \), just for fun. We get:

\[
i \sin \theta = i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \frac{i\theta^7}{7!} + \frac{i\theta^9}{9!} + \ldots \\
cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + \ldots
\]

Remember that \( i^2 = -1 \). So, we see \( i^3 = -i \), \( i^4 = 1 \), \( i^5 = i \), \( i^6 = -1 \), \( i^7 = -i \), \( i^8 = 1 \), etc. Notice the pattern? So we can write the above expressions as:

\[
i \sin \theta = i\theta + \frac{3i\theta^3}{3!} + \frac{5i\theta^5}{5!} + \frac{7i\theta^7}{7!} + \frac{9i\theta^9}{9!} + \ldots \\
\cos \theta = 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + \ldots
\]

Keeping up? Make sure you’re happy with what I’ve just done before moving on. We can write these as:

\[
i \sin \theta = (i\theta) + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^7}{7!} + \frac{(i\theta)^9}{9!} + \ldots \\
\cos \theta = 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^8}{8!} + \ldots
\]
Adding together the two expressions above (we’ll assume it’s OK to add together infinite sums of numbers), we get:

\[ \cos \theta + i \sin \theta = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \ldots \]

But if you look back at the Taylor series for \( e^x \), we see that this is just \( e^{i\theta} \), i.e.

\[ e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \ldots \]

So we’ve deduced that

\[ e^{i\theta} = \cos \theta + i \sin \theta \]

which is called Euler’s formula, (unsurprisingly, this is also in the formula booklet) and if we want, we can times through by \( r \):

\[ re^{i\theta} = r(\cos \theta + i \sin \theta). \]

So we can indeed write complex numbers as \( re^{i\theta} \).

**Another proof of De Moivre’s Theorem**

From Euler’s formula, which we described above, we know:

\[ \cos \theta + i \sin \theta = e^{i\theta}. \]

Taking both sides to the power \( n \) gives us:

\[ (\cos \theta + i \sin \theta)^n = (e^{i\theta})^n, \]

and remembering that taking a power of a power, we multiply the powers (i.e. \( a^{bc} = a^{bc} \)), we get:

\[ (\cos \theta + i \sin \theta)^n = e^{i(n\theta)}. \]

We use Euler’s formula again:

\[ e^{i(n\theta)} = \cos n\theta + i \sin n\theta, \]

and therefore we have De Moivre’s Theorem:

\[ (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta. \]

In fact, we’ve shown here that De Moivre is true for any number \( n \), not just positive integers, which is cool.
The bit you’ve been waiting for... Euler’s identity

Let’s use Euler’s formula again:

\[ e^{i\theta} = \cos \theta + i \sin \theta. \]

Put \( \theta = \pi \), and remember that we have to use RADIANS here. Then \( \sin \pi = 0 \) and \( \cos \pi = -1 \), so we conclude that

\[ e^{i\pi} = -1, \]

so taking the -1 over to the left hand side...

\[ e^{i\pi} + 1 = 0, \]

which is called Euler’s identity. Pretty cool, right? Who would’ve thought that the numbers 0, 1, \( \pi \), \( e \) and \( i \) would be so intimately related? This one equation uses the five most important numbers in maths (well, I think they’re the five most important numbers), and the three most important operations (add, times, powers), together with an equals sign. Amazing.